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19. **Eigenvalues of a 2×2 Hermitian matrix.** Let $A = \begin{pmatrix} a & b \\ b* & d \end{pmatrix}$ *where a, d are real numbers and b is some complex number.*

(a) *Find the eigenvalues and corresponding eigenvectors of A.*

A nonzero vector $\mathbf{v}$ and a scalar λ are an eigenvector and associated eigenvalue of an operator A if the following relation holds:

$$A\mathbf{v} = \lambda\mathbf{v} \tag{1}$$

We can rewrite that in the following way, where I denotes the identity operator:

$$A\mathbf{v} - \lambda\mathbf{v} = \mathbf{0}$$
$$(A - \lambda I)\mathbf{v} = \mathbf{0}$$

We see that the operator $A - \lambda I$ must be singular for it to send a non-trivial vector $\mathbf{v}$ to zero. An operator is singular if and only if its determinant vanishes. We may solve for the values of λ that produce a singular operator $A - \lambda I$; the resulting equation is called the *characteristic polynomial* of A:

$$\det(A - \lambda I) = 0 \tag{2}$$

To solve this particular problem, we substitute in the given 2×2 matrix for A, and the two dimensional identity matrix for I:

$$\det\left\{\begin{pmatrix} a & b \\ b* & d \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\} = \det\begin{pmatrix} a-\lambda & b \\ b* & d-\lambda \end{pmatrix} = (a-\lambda)(d-\lambda) - |b|^2 = 0$$

We find a quadratic equation for λ

$$\lambda^2 - (a+d)\lambda + (ad - |b|^2) = 0$$

whose solutions may be found using the quadratic equation:

$$\lambda = \frac{1}{2}\left((a+d) \pm \sqrt{(a+d)^2 - 4(ad - |b|^2)}\right)$$

$$\boxed{\lambda = \frac{1}{2}\left((a+d) \pm \sqrt{(a-d)^2 + 4|b|^2}\right)}$$

Note that the eigenvalues are both real-valued, as we expect for hermitian matrices, and that in the limiting case of a diagonal matrix ($b \to 0$), the eigenvalues are exactly the diagonal elements a and d.

Also, you might notice that the terms appearing in the characteristic polynomial suspiciously resemble the trace and determinant of the matrix. It turns out that the trace and determinant appear as coefficients in the characteristic polynomial of any matrix; in particular, the determinant is always the constant term, so the characteristic polynomial evaluated at zero will give the determinant. This can be seen by simply looking at the definition of the characteristic polynomial. Knowing the coefficients of the characteristic polynomial is equivalent to knowing the eigenvalues (the zeros of the polynomial); both are independent of the choice of basis. In some fields (e.g. Mechanical Engineering), the coefficients in the characteristic polynomial are known as 'the invariants' of the operator. The following hold for a general 2×2 matrix:

$$\boxed{\begin{array}{c}\lambda^2 - (\text{tr } A)\lambda + (\det A) = 0 \\ \lambda = \frac{1}{2}\left(\text{tr } A \pm \sqrt{(\text{tr } A)^2 - 4\det A}\right)\end{array}} \tag{3}$$

To find the eigen*vectors*, we take our knowledge of the allowed values of λ and use the relation $(A - \lambda I)\mathbf{v} = 0$ to find the corresponding vectors. We do this by looking for the null space of $A - \lambda I$:

$$\begin{pmatrix} a-\lambda & b \\ b* & d-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} (a-\lambda)v_1 + bv_2 \\ b^* v_1 + (d-\lambda)v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \boxed{v_2 = -\frac{a-\lambda}{b}v_1}$$

You may combine that with the requirement $|\mathbf{v}|^2 = v_1^2 + v_2^2 = 1$ if you'd like *normalized* eigenvectors.

(I find that the results are much more clear if the components of the eigenvector are written like this, in terms of the eigenvalue λ, rather than substituting in the rather cumbersome expression for λ in terms of the components of A; you're free to do the same.)

(b) *What are the conditions on a, b, d for the two eigenvalues to coincide?*

The expression to the right of the $\pm$ in our expression for λ must vanish:

$$(a-d)^2 + 4|b|^2 = 0$$

Both terms in this, $(a-d)^2$ and $|b|^2$, are positive, so there's no possibility of them cancelling out. (Remember that $a, d \in \mathbb{R}$ and $b \in \mathbb{C}$.) The condition for $\lambda_1 = \lambda_2$ becomes

$$\boxed{(a = d) \wedge (b = 0)}$$

In other words, for a 2×2 Hermitian matrix to have a single degenerate eigenvalue, it must be a diagonal matrix with the two diagonal entries equal, a multiple of the identity matrix:

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

20. **Resolvents.**

(a) *Find the resolvent of A.*

The resolvent of a matrix A is the inverse of that thing $(A - \lambda I)$ whose determinant we set to zero to find the characteristic polynomial. If I write $P_A(t) = A - tI$ then the characteristic polynomial is $p_A = \det P_A(t) = (t-\lambda_1)(t-\lambda_2)\cdots(t-\lambda_n)$ and the resolvent is $G_A(t) = (A - tI)^{-1}$. Without further ado, we may attack the problem at hand.

The inverse of a 2×2 matrix is

$$A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Using this, the resolvent is:

$$G_A(t) = (A - tI)^{-1} = \begin{pmatrix} a-t & b \\ b^* & d-t \end{pmatrix}^{-1} = \boxed{\frac{1}{\det(A - tI)}\begin{pmatrix} d-t & -b \\ -b^* & a-t \end{pmatrix}} = \frac{1}{p_A(t)}\begin{pmatrix} d-t & -b \\ -b^* & a-t \end{pmatrix}$$

(b) *Find the resolvent's singularities and the residues at these singularities.*

Clearly the resolvent has a singularity (a pole) wherever the characteristic polynomial has a zero, namely at the eigenvalues of A.

The residue of an mth-order poleof a function $f(z)$ at z_0 may be calculated as

$$\text{res}\ \{f(z); z_0\} = \lim_{z\to z_0} \left\{ \frac{1}{(m-1)!} \left(\frac{\partial}{\partial z}\right)^{m-1} (z-z_0)^m f(z) \right\} \tag{4}$$

Assuming that the two eigenvalues are distinct, then each pole will be a simple pole (a pole of order 1), so we have:

$$\text{res}\ \{G_A(z); \lambda_n\} = \lim_{z\to\lambda_n} \{(z-\lambda_n)G_A(z)\}$$

$$\text{res}\ \{G_A(z); \lambda_n\} = \lim_{z\to\lambda_n} \left\{(z-\lambda_n)\frac{1}{\det(A-zI)} \begin{pmatrix} d-z & -b \\ -b^* & a-z \end{pmatrix}\right\}$$

Remember that the eigenvalues λ_n are the roots of the characteristic polynomial $\det(A-\lambda I)$ so we may write $\det(A-zI) = (z-\lambda_1)(z-\lambda_2)$.

$$R_1 = \text{res}\ \{G_A(z); \lambda_1\} = \lim_{z\to\lambda_n} \left\{ \frac{\cancel{(z-\lambda_1)}}{\cancel{(z-\lambda_1)}(z-\lambda_2)} \begin{pmatrix} d-z & -b \\ -b^* & a-z \end{pmatrix}\right\} = \boxed{\frac{1}{\lambda_1-\lambda_2} \begin{pmatrix} d-\lambda_1 & -b \\ -b^* & a-\lambda_1 \end{pmatrix}}$$

(c) *Explain the relationship to A's eigenvectors and eigenvalues.*

What is the meaning of these matrices R_n that we get as the residues of the resolvent $G_A(z)$ evaluated at the eigenvalues λ_n of A? Let's consider *its* eigenvalues! Since I already used λ to designate eigenvalues of A, I'll use α to designate eigenvalues of R_n.

$$\det(R_1-\alpha I) = \det \frac{1}{\lambda_1-\lambda_2} \begin{pmatrix} d-\lambda_1-\alpha(\lambda_1-\lambda_2) & -b \\ -b^* & a-\lambda_1-\alpha(\lambda_1-\lambda_2) \end{pmatrix}$$

Let $\alpha' = \alpha(\lambda_1-\lambda_2)$.

$$= \frac{1}{(\lambda_1-\lambda_2)^2} \det \begin{pmatrix} d-\lambda_1-\alpha' & -b \\ -b^* & a-\lambda_1-\alpha' \end{pmatrix}$$

$$= \frac{1}{(\lambda_1-\lambda_2)^2} \left((d-\lambda_1-\alpha')(a-\lambda_1-\alpha') - |b|^2\right)$$

$$= \frac{1}{(\lambda_1-\lambda_2)^2} \left((a-(\lambda_1+\alpha'))\,(d-(\lambda_1+\alpha')) - |b|^2\right)$$

We recover the characteristic polynomial of A, but with a change of variable $\lambda \to (\alpha'+\lambda_n)$!

$$= \frac{1}{(\lambda_1-\lambda_2)^2}((\alpha'+\lambda_1)-\lambda_1)((\alpha'+\lambda_1)-\lambda_2)$$

$$= \frac{1}{(\lambda_1-\lambda_2)^2}(\alpha')(\alpha'-(\lambda_2-\lambda_1))$$

Replace α' with $\alpha(\lambda_1-\lambda_2)$:

$$= \frac{1}{(\lambda_1-\lambda_2)^2}\alpha(\lambda_1-\lambda_2)(\alpha(\lambda_1-\lambda_2)-(\lambda_2-\lambda_1))$$
$$= \alpha(\alpha-1)$$

The eigenvalues of the residue of the resolvent are exactly one and zero. Any operator which has only ones and zeros as eigenvalues is idempotent, meaning that applying it multiple times is equivalent to applying it only once, or $P^2 = P$. (In this case the result $R^2 = R$ pops out immediately if you substitute the operator R into its own characteristic equation, $\alpha(\alpha-1) = 0$.) Such operators are called projections. To find out onto what space the operator projects, we need to find the corresponding eigenvectors.

It turns out that the eigenvectors of R are the same as the eigenvectors of A. Earlier we found that the eigenvectors of a 2×2 hermitian matrix are multiples of $(1, -(a-\lambda)/b)$ where λ is the corresponding eigenvalue. We're interested in finding the vectors in the null space of $(R-\alpha I)$, where α is the eigenvector of R under consideration.

We'll first explore the eigenspace corresponding to $\alpha = 0$ by acting with $R - 0I = R$ on $\mathbf{v}_2$:

$$\frac{1}{\lambda_1-\lambda_2}\begin{pmatrix} d-\lambda_1 & -b \\ -b^* & a-\lambda_1 \end{pmatrix}\begin{pmatrix} 1 \\ -\frac{a-\lambda_1}{b} \end{pmatrix} = \frac{1}{\lambda_1-\lambda_2}\begin{pmatrix} d-\lambda_1+a-\lambda_2 \\ -b^*-(a-\lambda_1)(a-\lambda_2)/b \end{pmatrix}$$

To simplify this expression, we need to use two facts about matrices. First, the sum of the diagonal entries of a matrix equals the sum of the eigenvectors; this sum is called the trace of the matrix (or operator). Second, the product of the diagonal entries of a matrix equals the product of the eigenvalues; this product is the determinant of the matrix (or operator). In the case of our 2×2 hermitian matrix,

$$\mathrm{tr}\ A = a + d = \lambda_1 + \lambda_2$$
$$\det A = ad - |b|^2 = \lambda_1\lambda_2$$

Using these we may simplify our vector:

$$\begin{pmatrix} d-\lambda_1+a-\lambda_2 \\ -b^*-(a-\lambda_1)(a-\lambda_2)/b \end{pmatrix} = \begin{pmatrix} (a+d)-(\lambda_1+\lambda_2) \\ (-|b|^2-a^2+(\lambda_1+\lambda_2)a-\lambda_1\lambda_2)/b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So we find that $R_1\mathbf{v}_2 = 0$, where $A\mathbf{v}_2 = \lambda_2\mathbf{v}_2$, and R_1 is the residue of the resolvent of A evaluated at λ_1. Through a similar argument we can show that $R_1\mathbf{v}_1 = \mathbf{v}_1$, i.e. that R has the same eigenvectors as A. **We find that the residue R_n of the resolvent of A at λ_n (an eigenvalue of the hermitian 2×2 matrix A with corresponding eigenvector $\mathbf{v}_n$) is a projection operator which projects onto the space spanned by that eigenvector.**

This is one result that is much easier to prove in the abstract, coordinate-free formalism. Following [2], suppose A is a hermitian operator (on a space of arbitrary dimension); then we know that we can form a basis of eigenvectors. Using Dirac bra-ket notation, suppose the eigenvectors are $|\mathbf{v}\rangle$.

The "outer product" $|v\rangle\langle v|$ of a vector $|v\rangle$ with itself forms a projection operator onto the space spanned by that vector. To see this, try it out on a test vector $|x\rangle$:

$$(|\mathbf{v}\rangle\langle\mathbf{v}|)|\mathbf{x}\rangle = |\mathbf{v}\rangle(\langle\mathbf{v}|\mathbf{x}\rangle)$$

The inner product $\langle\mathbf{v}|\mathbf{x}\rangle$ is a scalar that may be interpreted as the magnitude of the projection of $\mathbf{x}$ onto $\mathbf{v}$. The additional $\mathbf{v}$ ket on the left provides that the result is a ket in the direction of $\mathbf{v}$. Hence the combination $|\mathbf{v}\rangle\langle\mathbf{v}|$ works as a projection operator.

Taking $\{|\mathbf{v}_n\rangle\}$ to be an orthonormal basis of eigenvectors of A, we can write the identity operator as a sum of projection operators:

$$I = \sum_{i=0}^{n} |\mathbf{v_i}\rangle\langle\mathbf{v_i}| \tag{5}$$

Apply (from the left) the operator A to each side of this identity. Because of the eigenvalue/vector relation $A|\mathbf{v_i}\rangle = \lambda_i|\mathbf{v_i}\rangle$ we get

$$A = \sum_{i=0}^{n} \lambda_i |\mathbf{v_i}\rangle\langle\mathbf{v_i}| \tag{6}$$

We wish to write the resolvent $G_A(\lambda) = (A - \lambda I)^{-1}$ in a similar format. Combining the above two results, we can write $A - \lambda I$ as:

$$A - \lambda I = \sum_{i=0}^{n} (\lambda_i - \lambda) |\mathbf{v_i}\rangle\langle\mathbf{v_i}|$$

Intuitively, to invert an operator that is written as a sum of scaled projections along orthogonal directions, we need only undo the scalings along each projection in order to invert the operator. Formally, we have written the operator as a diagonal matrix, with entries $(\lambda_i - \lambda)$ along the diagonal, and we know that to find the inverse of a diagonal matrix we need only invert each (scalar) element along the diagonal:

$$(A - \lambda I)^{-1} = \sum_{i=0}^{n} (\lambda_i - \lambda)^{-1} |\mathbf{v_i}\rangle\langle\mathbf{v_i}|$$

To prove that this is the inverse, write $(A - \lambda I)(A - \lambda I)^{-1}$, substitute in the summation forms, and mltiply. By the assumption that $\{|\mathbf{v}_n\rangle\}$ is an orthonormal basis, $\langle\mathbf{v_i}|\mathbf{v_j}\rangle = \delta_{ij}$ and the product of sums turns into a single sum, the scalars in front of the projections cancel out, and we're left with an expansion for the identity operator. So, we've succeeded in finding the resolvent as a sum of projections on normalized eigenvectors of A:

$$G_A(t) = \sum_{i=0}^{n} \frac{|\mathbf{v_i}\rangle\langle\mathbf{v_i}|}{(\lambda_i - t)} \tag{7}$$

(Notice that $G_A(0)$ has the form of the eigenvalue expansion of a Green's function as in, for instance, page 94 of [1].)

Now consider the residues R_n of $G_A(\lambda)$ at eigenvalues λ_n of A. The residue operation selects those terms in the sum which have $(\lambda_n - t)$ in the denominator. These are all simple poles, so we multiply by $(\lambda_n - t)$; we would then substitute $t \to \lambda_n$, but there are no remaining occurrances of t. We're left with:

$$R_n = \mathrm{res}\{G_A(t);\, t = \lambda_n\} = \sum_{\{i:\lambda_i = \lambda_n\}} |\mathbf{v_i}\rangle\langle\mathbf{v_i}| \tag{8}$$

This is a direct, general expression of what we saw before: the residues of the resolvent are projection operators onto the eigenspaces associated with the eigenvalues where we took the residues.

21. **Function spaces.**

Let V be the space of all complex-valued functions of a real variable with period 2π.

A vector space is specified by a set V, a scalar field F, an operation of 'addition' ($+ : V \times V \mapsto V$) between any two elements of the set, and an operation of 'scalar multiplication' that combines a vector with a scalar to get another vector ($\cdot : V \times F \mapsto V$). Here our set is the set of 2π-periodic complex-valued functions of a real variable:

$$V = \{f : R \mapsto C \mid \forall z \in \mathbb{R}, f(z + 2\pi) = f(z)\}$$

We define addition and scalar multiplication of functions:

$$f_1 + f_2 : \forall f_1, f_2 \in V, \forall x \in \mathbb{R}\, (f_1 + f_2)(x) = f_1(x) + f_2(x)$$

$$\alpha f : \forall \alpha \in \mathbb{C}, \forall f \in V, \forall x \in \mathbb{R},\ (\alpha f)(x) = \alpha(f(x))$$

To maintain closure under scalar multiplication, we see that the scalar field must be the complex numbers.

(a) *Find a basis for this space. What is its dimension?*

From basic Fourier analysis, we know that any periodic function can be written as a sum of complex exponentials. That set of complex exponentials is our basis:

$$\boxed{\beta = \{f \mid f(t) = e^{itn},\ n \in \mathbb{Z}\}}$$

To prove that β is a basis for V, we must show that any element of V may be written as a linear combination of elements of β, and that no element of β may be written as a linear combination of other elements of β. These are results we know from analysis.

The dimension of a space is the cardinality of its basis, so the dimension of V is infinite.

$$\boxed{\dim V = |\beta| = \infty}$$

(b) *For what values of λ are there solutions in V to $(\partial/\partial t)^2\psi(t) = \lambda\psi(t)$?*

Differentiation is a linear operation on functions. Suppose D is the operator that takes the derivative of a function with respect to its first argument; we can write $f'(x)$ as $(Df)(x)$. We wish to find λ such that $D^2 f = \lambda f$, or, in other words, solve the eigenvalue problem for D^2.

As with any other problem involving the characterization of operators, we need only examine the effect of the operator D^2 on the basis vectors. Let $\{\psi_n\}$ be the set of basis vectors, with $\psi_n(t) = \exp\{itn\}$.

$$(D\psi_n)(x) = \frac{\partial}{\partial x} e^{inx} = ine^{inx} = (in\psi_n)(x)$$
$$D^2\psi_n = -n^2\psi_n$$
$$\boxed{\lambda = -n^2,\ n \in \mathbb{Z}}$$

We see that the allowed eigenvalues λ are $\lambda = -n^2$ for integral n.

(c) *Find a basis of solutions for each such value of λ.*

For any given λ, there are solutions in our space only if there is an integer n such that $\lambda = -n^2$. In this case, there are one or two eigenvectors, namely ψ_n where $n \in \{\pm\sqrt{\lambda}\}$. Either of these eigenvectors will solve the original expression $(d/dx)^2\psi(x) = \lambda\psi(x)$, as will any linear combination of them; this set of two vectors is a basis for the space of solutions to that equation for given $\lambda \neq 0$.

$$\boxed{\beta = \begin{cases} \{\exp(it\sqrt{\lambda}), \exp(-it\sqrt{\lambda})\} & \text{if } \exists n \in \mathbb{Z}, -n^2 = \lambda \\ \emptyset & \text{otherwise} \end{cases}}$$

The solutions to the equation are linear combinations of these basis vectors:

$$f(x) = A\exp(-it\sqrt{\lambda}) + B\exp(it\sqrt{\lambda}) \text{ where } A, B \in \mathbb{C}$$

References

[1] Leonard Eyges. *The Classical Electromagnetic Field.* Dover, 1972.

[2] Ulrich Gerlach. *Linear Mathematics in Infinite Dimensions: Signals, Boundary Values Problems, and Special Functions.* 2004. Available freely online at `http://www.math.ohio-state.edu/~gerlach/math/BVtypset/`.