PHY404 Linear Spaces Spring 2006

Final Exam Wednesday May 3 2006 9:30-10:45am

1 Define the creation-annihilation operators by

$$a^\dagger|n>=\sqrt{(n+1)}|n+1> \text{ for } n=0,1,\cdots,\ a|n>=\sqrt{n}|n-1> \text{ for } n=1,2\cdots, \tag{1}$$

and $a|0>=0$. Here the collection of vectors $|n>, n=0,1,\cdots$ is an orthonormal basis. For *any* complex number z, find an eigenvector for a with eigenvalue z, as a linear combination $\sum_{n=0}^{\infty} c_n(z)|n>$. Find the length of the eigenvector by evaluating the sum $\sum_0^\infty |c_n(z)|^2$ and use that to find an eigenvector of unit length.

2 Consider the matrix

$$A=\begin{pmatrix}1 & -i & 0\\ i & 1 & 0\\ 0 & 0 & -2\end{pmatrix} \tag{2}$$

2.1 What are its eigenvalues and eigenvectors?

2.2 Find its resolvent $R(z)=(A-z)^{-1}$ and verify that the positions of the poles are the eigenvalues.

Solutions

1. If we put

$$a \sum_{n=0}^{\infty} c_n(z)|n >= z \sum_{n=0}^{\infty} c_n(z)|n > \tag{3}$$

we get

$$\begin{aligned} L.H.S. &= \sum_{1}^{\infty} \sqrt{n}\ c_n(z)|n-1> \\ &= \sum_{0}^{\infty} \sqrt{[n+1]} c_{n+1}(z)|n> \\ &= z \sum_{0}^{\infty} c_n(z)|n> . \end{aligned} \tag{4}$$

So

$$\sqrt{[n+1]} c_{n+1}(z) = z c_n(z). \tag{5}$$

Or,

$$\begin{aligned} c_n(z) &= \frac{z}{\sqrt{n}} c_{n-1}(z) \\ &= \frac{z}{\sqrt{n}} \frac{z}{\sqrt{[n-1]}} c_{n-2}(z) \\ &= \cdots \\ &= \frac{z}{\sqrt{n}} \frac{z}{\sqrt{[n-1]}} \cdots \frac{z}{\sqrt{1}} c_0(z) \end{aligned} \tag{6}$$

That is,

$$c_n(z) = \frac{z^n}{\sqrt{[n!]}} c_0(z). \tag{7}$$

The square of the length of the vector is

$$\sum_{0}^{\infty} |c_n(z)|^2 = |c_0(z)|^2 \sum_{0}^{\infty} \frac{|z|^{2n}}{n!} = |c_0(z)|^2 e^{|z|^2} \tag{8}$$

Thus the length is finite for *any* complex number z.

Setting the length to one gives

$$c_0(z) = e^{-\frac{1}{2}|z|^2}. \tag{9}$$

Thus any complex number is an eigenvalue of a with normalized eigenvector

$$e^{-\frac{1}{2}|z|^2} \sum_{0}^{\infty} \frac{z^n}{\sqrt{[n!]}} |n> . \tag{10}$$

2. The characteristic polynomial of the matrix is

$$\begin{vmatrix} 1-z & -i & 0 \\ i & 1-z & 0 \\ 0 & 0 & -2-z \end{vmatrix} = -(z+2)[(1-z)^2-1] = -(z+2)[z^2-2z] = -z(z-2)(z+2). \tag{11}$$

Thus the eigenvalues are $0, 2, -2$. The eigenvector corresponding to -2 is easiest to find

$$\begin{pmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 0 \Rightarrow u_1 = u_2 = 0. \tag{12}$$

For the eigenvalue 0,

$$\begin{pmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 0 \Rightarrow \; u_1 = iu_2, \; u_3 = 0. \tag{13}$$

And for the eigenvalue 2

$$\begin{pmatrix} -1 & -i & 0 \\ i & -1 & 0 \\ 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 0 \Rightarrow \; u_1 = -iu_2, \; u_3 = 0. \tag{14}$$

Choosing the length of each eigenvector to be one gives

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \; \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}, \; \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \tag{15}$$

respectively as the eigenvectors of the eigenvalues $-2, 0, 2$.

The resolvent is the inverse

$$R(z) = \begin{pmatrix} 1-z & -i & 0 \\ i & 1-z & 0 \\ 0 & 0 & -2-z \end{pmatrix}^{-1} \tag{16}$$

Because it is block diagonal we can break the matrix up into a two by two matrix

$$r(z) = \begin{pmatrix} 1-z & -i \\ i & 1-z \end{pmatrix}^{-1} \tag{17}$$

and inverting just the number $-2-z$. Now

$$r(z) = \frac{1}{z(z-2)} \begin{pmatrix} 1-z & i \\ -i & 1-z \end{pmatrix} \tag{18}$$

Thus

$$R(z) = \begin{pmatrix} \frac{1-z}{z(z-2)} & \frac{i}{z(z-2)} & 0 \\ -\frac{i}{z(z-2)} & \frac{1-z}{z(z-2)} & 0 \\ 0 & 0 & -\frac{1}{2+z} \end{pmatrix} \tag{19}$$

The singularities are at the points $z = -2, 0, 2$ which are the eigenvalues of the matrix A.