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Problem number four on the Prelim exam covers material from this course. I've collected this question from all of the old prelim exams that I have here for you to use as practice problems for use while studying for the Physics 402 final. Solving these problems should enable you to ace both this course and the relevant question on the prelim! (-: –TF

1 January 2006

Let $x > 0$ be a random variable with probability density $p(x) = \exp\{-x\}$.

(a) Find the probability for x to deviate from its mean by more than two standard deviations.

(b) Compare your result with the bound obtained from the Chebyshev inequality.

2 August 2005

Two discrete random variables x and y have the following probability distributions:

$$p_x(k) = \begin{cases} A & k \in \{0, 1, 2\} \\ 0 & k \geq 3 \end{cases}$$

$$p_y(k) = \frac{2B^k}{k!}, k \in \{0, 1, 2, \cdots\}$$

where A and B are normalization constants. A third random variable z is defined as $z = x + y$.

(a) Derive the generating function of z.

(b) Find the average and variance of z.

(c) Let z_i be N independent variables all distributed like z. Find the distribution of their sum $S_N = \sum_{i=1}^{N} z_i$ as $N \to \infty$.

3 January 2005

I don't have a copy of this problem.

4 August 2004

Let η_j for $j \in \{1, 2, \cdots, n\}$ be n independent and identically distributed continuous random variables. They are all uniformly distributed in the interval $[0, 1]$ with probability density $p(\eta) = 1$ for $0 \leq \eta \leq 1$ and $p(\eta) = 0$ otherwise.

(a) Find the probability that all n random variables η_j are larger than y for y in the interval $[0, 1]$.

(b) Find the expectation value of x, the value of the smallest of all the η_j, i.e., $x = \min\{\eta_j, j = 1, 2, \cdots, n\}$.

(c) Find the variance of x.

5 January 2004

(a) Given two independent random variables ξ_1, ξ_2, uniformly distributed in the range $[0,1]$, find the probability distribution of $\xi = \xi_1 + \xi_2$.

(b) Determine the mean and variance of ξ.

6 September 2003

(a) Let ξ be a Gaussian random variable of mean zero and standard deviation σ. Find the probability density function of $\eta = \xi^2$.

(b) Let $\xi_1, \xi_2, \cdots, \xi_n$ be Gaussian random variables, each of mean zero and standard deviation σ. Find the probability density function of the variable $\eta_n = \frac{1}{n}\sum_{i=1} n\eta_i{}^2$.

7 January 2003

(a) Find the probability distribution of $aX + bY$ for real numbers a and b.

(b) Show that $U = aX + bY$ and $V = cX + dY$ are independent random variables if and only if the vectors (a, b) and (c, d) are orthogonal.

8 September 2002

Let ξ be a random variable with positive integer values, with probability $p[\xi = n] = p_n$. Let $F_\eta(z) = \sum_{n=1}^{\infty} p_n z^n$ be the generating function for ξ. Let $\sigma(\xi)$ be the random variable which is 0 if ξ is even, and 1 if ξ is odd.

(a) Show that $F_\eta(-1) = F_{\sigma(\eta)}(-1)$.

(b) Find expressions for $P\{\sigma(\eta) = 0\}$ and $P\{\sigma(\eta) = 1\}$ in terms of $F_\eta(-1)$.

(c) Now suppose that η_k is a sequence of independent copies of a random variable with $0 < p_0$, $p_1 < 1$, and $p_j = 0$ for $j > 1$. Show that

$$\lim_{N\to\infty} P\left\{\sigma\left(\sum_{k=1}^{N} \xi_k\right) = 0\right\} = \lim_{N\to\infty} P\left\{\sigma\left(\sum_{k=1}^{N} \xi_k\right) = 1\right\} = \frac{1}{2}$$

i.e., no matter what p_0 and p_1 are, the probabilities that the sum of a number of these random variables is even or odd become equal as the number of terms becomes large.

9 January 2002

A system has N possible states. At each second, the system goes to a new state; if it is in state j, it goes to state i with probability p_{ij}, where $p_{ij} = p_{ji} > 0$.

(a) Let $p^{(n)} = \left(p_1^{(n)} \cdots p_N^{(n)}\right)$ be the probability distribution of the staes at time n. Show that if the initial distribution is uniform, i.e., $p_i^{(0)} = \frac{1}{N}$, then $p_i^{(n)} = p_i^{(0)}$.

(b) Show that, in general, $p^{(n)} = P^n p^{(0)}$, where P is the matrix P_{ij}.

(c) Let λ be any eigenvalue of P, with normalized eigenvector $\hat{\mathbf{v}} = (v_1, \cdots, v_n)$. Suppose that λ_m is the largest eigenvalue of P. Show that $|\lambda| \leq \sum_{i,j=1}^{N} |v_i| P_{ij} |v_j| \leq \lambda_m$, and the left inequality is strict ($<$) unless v_i has the same sign for all i.

(d) Show that there is only one eigenvalue whose eigenvector has this property (i.e., all of its components have the same sign), so $\lambda_m = 1$.

(e) Show that for any initial probability distribution $p^{(0)}$, as n approaches infinity, $p^{(n)}$ approaches the uniform distribution $(1/N, \cdots, 1/N)$.

10 September 2001

(a) Suppose a coin toss has a probability $p \leq \left(\frac{1}{2} - \delta\right)$ of heads. What does Chebyschev's inequality imply about the probability that out of N tosses of this coin, at least $N(1-\delta)/2$ are heads? How large does N have to be to insure that this probability is less than ϵ?

(b) If A and B are events, let $P(A|B)$ represent the conditional probability that A occurs, given the occurance of B. Suppose that A and A' are mutually exclusive. Express $P(A|B)/P(A'|B)$ in terms of $P(A|B)$, $P(B|A')$, $P(A)$, and $P(A')$. If $\ A$ is defined as the event that A does not happen, note that $P(A'|B) \leq P(\ A|B)$.

(c) How could you decide that a coin is probably fair (i.e., the probability of heads is acceptable close to 1/2) by tossing it a sufficient number of times? Discuss this using (a) and (b) and determine how many tosses are needed to get a given degree of certainty. Does probability theory alone answer the question?

11 January 2001

Who says the prelim can't reflect current political events?

(a) Derive Chebyschev's inequality: If X is a random variable, $E\{X\}$ is its expectation, and $D\{X\}$ its variance, then $P\{|X - E\{X\}| > \epsilon\} \leq \epsilon^2 D\{X\}$.

(b) Two candidates A and B run in an election. The result is to be predicted by asking a random sample of voters, of size N, how they voted.

Suppose the proportion of all voters for A is p with $1-p$ for B.

How large must N be, according to Chebyschev's inequality, so that if the sample has k out of N votes for A, then $P\{k/N - p| > 0.01\} < 0.1$?

(c) Suppose there are six million votes cast, and ten percent of them are rejected by the counting machine (and the probability of rejection is independent of the choice of the voter.) Suppose that 300 more votes are cast for A than B. Estimate the probability that not counting the rejected votes changes the result using the integral of the normal distribution (with mean zero and unit variance) from $-\infty$ to x as shown below. (*on the prelim there was a little graph of the normal cdf. not included here.-TF*)

12 September 2000

Suppose that ξ is a random variable.

(a) Define the characteristic function $f_\xi(t)$ for ξ.

(b) Express the mean and variance of ξ in terms of f_ξ.

(c) Suppose that η and ξ are independent random variables with the same probability distribution. Show that if $\xi + \eta$ has the same probability distribution as 2ξ, then ξ is deterministic, i.e., it takes a certain value with probability one.

13 January 2000

(a) N balls are placed in an urn, with equal probability of being black or white. What is the probability $p(m)$ that exactly m of the N balls are black?

(b) Suppose a ball is drawn at random from the urn as prepared in (a), and then returned. If this drawing is repeated k times, what is the probability that all of the k balls drawn are black?

(c) If this happens, i.e., k out of k balls drawn are black, what is the probability that all N balls in the urn are black? Show that this probability approaches 1 as k goes to infinity.