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16. *Let ξ_1, ξ_2 be two independent Gaussian random variables, both of zero mean and unit variance. What are the probability densities of the variables $\rho = (1/2)\left(\xi_1^2 + \xi_2^2\right)$ and $\theta = \arctan \frac{\xi_2}{\xi_1}$?*

$$\boxed{\begin{array}{l} p_\rho(x) = \begin{cases} e^{-x} & 0 \le x < \infty \\ 0 & \text{otherwise} \end{cases} \\ \\ p_\theta(x) = \begin{cases} 1/2\pi & 0 \le x < 2\pi \\ 0 & \text{otherwise} \end{cases} \end{array}} \tag{1}$$

17. *Let η_1, η_2 be independent random variables with uniform (constant) probability density in the range $[0, 1]$. Find changes of variables $\xi_1 = f_1(\eta_1, \eta_2)$, $\xi_2 = f_2(\eta_1, \eta_2)$ which produces a pair of independent Gaussian random variables of zero mean and unit variance.*

    This is known as the **Box-Muller method**.

$$\boxed{\begin{array}{l} \xi_1 = f_1(\eta_1, \eta_2) = \sqrt{-2\log\eta_1}\cos(2\pi\eta_2) \\ \xi_2 = f_2(\eta_1, \eta_2) = \sqrt{-2\log\eta_1}\sin(2\pi\eta_2) \end{array}} \tag{2}$$

18. *Generate a data set of a thousand values of a Gaussian random variable of zero mean and unit standard deviation, starting from the built-in function in most computer languages that produces uniform variables in the range $[0, 1]$. Plot a histogram of the relative frequencies of this data set against the predicted probability density function*

```
% Generate the uniformly distributed random numbers
eta1 = rand(500,1);
eta2 = rand(500,1);

% Use the Box-Muller transform to get normally distributed numbers
xi1 = sqrt(-2 * log(eta1)) .* cos(2*pi*eta2);
xi2 = sqrt(-2 * log(eta1)) .* sin(2*pi*eta2);

% Produce a histogram
[rf, x] = hist([xi1; xi2], 50);

% Compute the theoretical distribution from the normal pdf
trf = (1/sqrt(2*pi)) * exp(-x.^2 /2);

% Normalize
rf = rf / sum(rf);
trf = trf / sum(trf);

% Produce the figure
h = figure;
plot(x, rf, '-x', x, trf, '-');
legend('experimental','theoretical');
title('generation of 1000 random deviates using box-muller method');
print(h, '-dpng', 'boxmuller.png');
close all;
```

Here's a plot of the sample distribution and the theoretical distribution: