Tools and Concepts Involved in the Representation of Rotation Groups in Physics

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Abstract

The purpose of this paper is to provide in a written format the topics covered in the Kapitza Lecture held on October 26, 2019. As such, it will follow closely to the material covered in chapter IV.1 of the book *Group Theory in a Nutshell for Physicists* for which the lecture was based on. Important topics include: how to construct irreducible representations of the special orthogonal group of dimension N, an explanation of dual tensors and their significance in certain groups, formulation and application of the anti-symmetry symbol.

1 SO(N) and General Concepts

An $N \times N$ matrix R is a member of the special orthogonal rotation group SO(N) if and only if (1) and (2) are true concerning R:

$$R^T R = 1 \tag{1}$$

$$det(R) = 1 \tag{2}$$

Irreducible representations in the N^{th} dimension can themselves be represented as a vector. As such, it is important to define how vectors behave under rotation. If we consider a vector V being rotated to V' then we say for i, j ranging from 1 to N, the components of V transform as follows:

$$V^i \rightarrow V'^i = R^{ij} V^j \tag{3}$$

In general, a vector will not work. Namely, if we try to create irreducible representations of dimensions higher than N, we will require a more complicated object. This object is called a tensor (essentially a generalized matrix). For a tensor T, we will treat its definition as an object that transforms under rotation in the following way:

$$T^{ij} = T'^{ij} = R^{ik}R^{jl}T^{kl} \tag{4}$$

where i, j range from 1 to N and l, k are summation indices ranging from 1 to N. Although in general tensors can have as many indices, for most purposes in physics, tensors with two indices (called a rank 2 tensor to indicate this) and lower are used. Rank 1 tensors are equivalent to vectors and rank 0 tensors are equivalent to scalars.

2 Representation Theory

A matrix representation of a group G means that each element of the group has a corresponding matrix that preserves the group operation through matrix multiplication. Mathematically, for an $N \times N$ matrix representation of G with group elements g_1, g_2 this looks like:

$$M(g_1)M(g_2) = M(g_1 g_2) \tag{5}$$

where $M(g_i)$ is the $M \times M$ matrix associated with the group element g_i. If we are dealing with a rank 2 tensor in 3 dimensions, we can represent the 9 components of the tensor in a 9×1 matrix as so:

$$T = \begin{bmatrix} T^{11} \\ T^{12} \\ ... \\ T^{33} \end{bmatrix}$$

For a 3×3 matrix R associated with a rotation, we can formulate a 9×9 matrix that can act on our column matrix (above). If we think about the resultant matrix we will see that, for each new T'^{ij}, it will be a linear combination of all the components in the original column matrix. However, all the objects in the column matrix were arbitrary values so the real thing of interest is our 9×9 rotation matrix that takes a 3×3 matrix (i.e. our tensor) and transforms it. Thinking about it this way makes it clear that this 9×9 matrix is the representation we wanted. To see this mathematically, we can compare

$$T^{ij} \to T''^{ij} = R_2^{ik}R_2^{jl}T'^{kl} = R_2^{ik}R_1^{km}R_2^{jl}R_1^{ln}T^{mn} = (R_2R_1)^{im}(R_2R_1)^{jn}T^{mn} \tag{6}$$

to equation (5) to see explicitly that $M(R_2)M(R_1) = M(R_2R_1)$ i.e. this meets our definition of a 9×9 matrix representation of the rotations in SO(3).

Determining Irreducible Representations When considering whether or not a representation is irreducible, we must try to find the collection of elements of the representation that can only be transformed into one another (kind of like a subgroup). The physicist way to do this is to first determine the common collections that almost always appear. One common collection of such elements is the anti-symmetric tensors. Typically denoted A, an anti-symmetric tensor is a tensor whose constituent parts can be written as:

$$A^{ij} = T^{ij} - T^{ji} \tag{7}$$

For such a tensor, it is simple to verify that $A^{ij} = -A^{ji}$ and under rotation $A'^{ij} = -A'^{ji}$. So we can see that our anti-symmetric tensors only can be turned into other anti-symmetric tensors. If we are interested in the number of possible unique constituent parts for a given tensor A, simple combinatorics lead us to the conclusion this number is, for a tensor in dimension N, $\frac{N(N-1)}{2}$.

The next common collection is the symmetric tensors, a counterpart to the anti-symmetric tensors. Typically these are denoted S and have constituent parts that can be written as:

$$S^{ij} = T^{ij} + T^{ji} \tag{8}$$

from which it is evident that $S^{ij} = S^{ji}$ and hence $S'^{ij} = S'^{ji}$. For a dimension N, the number of possible unique constituent parts is $\frac{N(N-1)}{2} + N$.

Using these facts, we can see that for SO(3) we can break our 9×9 matrices into 3×3 anti-symmetric matrices and 6×6 symmetric matrices. Given that the qualities of symmentry and anti-symmetry for a tensor are mutually exclusive and $6 + 3 = 9$, we can conclude that we aren't missing any other collections.[1] However, it turns out that the symmetric tensor collection is not irreducible. For a given rotation, the diagonal elements of S are sent to one another, meaning that the trace remains unaltered. As such, symmetric matrices can be broken down into traceless symmetric matrices and diagonal matrices with a certain trace. It is clear that the diagonal matrices of a given trace will have a dimensionality of 1, so we find that in general the traceless symmetric matrices will be of dimension $\frac{N(N-1)}{2} + N - 1$. For SO(3), this means that our 9×9 matrix representation can be decomposed into 1×1, 3×3, and 5×5 matrices.

Dimensions of the SO(3) Irreducible Representations Suppose we are interested in counting the total number of independent components (i.e. dimensionality) of a symmetric tensor representation of SO(3) with N indices (e.g. $S^{i_1 i_2 \cdots i_N}$). Then each index can only take one of three values and, since it is

[1] For a general N with a rank-2 tensor representation (of dimension N^2), since $\frac{N(N-1)}{2} + \frac{N(N-1)}{2} + N = N^2$, this line of reasoning will hold in general for $N \times N$ matrix representations.

symmetric, only the total number of each value matters (e.g. $S^{klmklmklm} = S^{kkklllmmm}$). The total number of possibly unique tensor components is:

$$\frac{(N+1)(N+2)}{2} \tag{9}$$

Considering, however, that we are interested only in traceless symmetric tensors, we will say that $\delta^{i_1 i_2} S^{i_1 i_2 \ldots i_N} = 0$. This determines two of the indices, leaving us with $N-2$ undetermined indices. Using (9), we can see that this leaves us with $\frac{N(N-1)}{2}$ components for traceless symmetric matrices. By assuming tracelessness, we are imposing this number of conditions meaning that we must remove this number from our total number of possibly unique components. This gives us $\frac{(N+1)(N+2)}{2} - \frac{N(N-1)}{2} = 2N+1$, which is our dimensionality of the irreducible representation under consideration.

3 Anti-symmetric Symbol and Kronecker Delta Under Rotation

Understanding how the anti-symmetric symbol and kronecker delta behave under rotation will provide much utility throughout the rest of this paper. The primary thing to note about both is that they are invariant under rotation.

Kronecker Delta To see why the kronecker delta is invariant, we can refer explicitly to the invariance of the trace under rotation, a fact that we glossed over in the last section. If we consider a transformation on an element of the trace of a symmetric rank 2 tensor, we have:

$$S^{ii} \to S'^{ii} = R^{ik} R^{il} S^{kl} = (R^T)^{ki} R^{il} S^{kl} \tag{10}$$

Referring to (1), if we are working in SO(N), then we can say

$$(R^T)^{ki} R^{il} S^{kl} = (R^{-1})^{ki} R^{il} S^{kl} = \delta^{kl} S^{kl} = S^{kk} \tag{11}$$

Considering that we can write $S'^{ii} = \delta^{ij} S'^{ij}$ and $S^{kk} = \delta^{kl} S^{kl}$ then we see that the preservation of the trace as shown in (10) and (11) gives us the fact that a kronecker delta under rotation will merely be sent to another kronecker delta i.e. it is also invariant.

Anti-symmetric Symbol The anti-symmetric symbol in N-dimensions is denoted with an ϵ containing N indices and obeys (12) and (13):

$$\epsilon^{\ldots i \ldots j \ldots} = -\epsilon^{\ldots j \ldots i \ldots} \tag{12}$$

$$\epsilon^{12...N} = 1 \tag{13}$$

For an arbitrary matrix M, we can express the determinant using the anti-symmetric symbol as follows:

$$\epsilon^{ij...n} det(M) = M^{ip} M^{jq} ... M^{nm} \epsilon^{pq...m} \tag{14}$$

if we consider that in SO(N) we are dealing with rotation matrices R that have a determinant of 1, then we can see that the left hand side of (14) is just an anti-symmetric symbol. So, similar to the kronecker delta, we find that anti-symmetric symbols are transformed under rotation into other anti-symmetric symbols.

4 Dual Tensors

For an anti-symmetric tensor A with m indices, we can multiply it with an anti-symmetric symbol with n indices and (assuming $n \geq m$) this yields another anti-symmetric tensor scaled by n with $n - m$ indices, typically denoted B. In this case, A and B are considered "dual" to each other and transform like each other.

For a rank 2 tensor in SO(3), we can see that for the anti-symmetric 3×3 matrices, these are dual to a rank 1 pseudotensor ($\frac{1}{2}\epsilon^{ijk} A^{ij} = B^k$). Since rank 1 tensors are just vectors, this tells us that our 3 dimensional irreducible representation transforms like a vector i.e. it is equivalent to a vector representation (like the Euler vectors). This fact has particular consequences, principally that we only have to deal with the traceless symmetric part of a tensor. It turns out that when we decompose a tensor T of N indices into its symmetric and anti-symmetric parts (denoted $T^{\{ij\}...n}$ and $T^{[ij]...n}$ respectively for the pair of indices i, j), the anti-symmetric part will always behave like an anti-symmetric tensor of $N - 1$ indices. Applying this observation N times allows us to treat any anti-symmetric part of a tensor as if it were a rank 1 tensor, leaving us with the symmetric component as the only part of real consequence. Evidently, we can decompose the symmetric tensor again into a traceless part and the tensor used to subtract out the trace. As we covered in section 2, the trace does not transform so the traceless symmetric part of the tensor ends up being the only part of interest.

Self-Dual and Antiself-Dual In general for SO(2N), we can define a tensor B dual to an anti-symmetric tensor A carrying N indices as follows:

$$B^{i_1 i_2 ... i_N} = \frac{1}{N!} A^{i_{N+1} i_{N+2} ... i_{2N}} \tag{15}$$

Then for such tensors A and B over the same N indices, we can define a self-dual tensor (denoted T_+) as $A + B$ and an antiself-dual tensor (denoted T_-) as $A - B$. The significance of these self-dual and antiself-dual tensors is that they correspond to irreducible representations. Considering that in SO(2N) an anti-symmetric matrix A with N indices will have $\frac{(2N)!}{(N!)^2}$ components, we can see that each self-dual and antiself-dual tensor will each have half this number of components (recall this is interchangeable with dimensionality for tensors).

5 Contraction of Indices

This section is merely to provide information on a convention used in the book which may be important to understanding the content of other papers based off of later sections of the book.

For a tensor where two indices are equal (e.g. $T^{ijk\dots jl}$), it will transform the same as a tensor with the indices removed (e.g. $T^{ik\dots l}$). To convince oneself, merely consider that $R^{jq}R^{jp} = \delta^{qp}$. So in a rotation of a tensor with two equal indices (think of one represented similar to in (4)), the two rotational components for each of the indices can be combined into a kronecker delta, which essentially functions as a 1 since we are summing over the indices. Since these rotational components are removed, the rotation looks the same as if we were considering the rotation of a tensor without them to begin with.

References

[1] A. Zee. *Group Theory in a Nutshell for Physicists.* Princeton University Press, 2016.

[2] Weisstein, Eric W. "Dual Tensor." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/DualTensor.html