The source of curvature

Fatima Zaidouni

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Abstract

In this paper, we explore how the local spacetime curvature is related to the matter energy density in order to motivate Einstein's equation and gain an intuition about it. We start by looking at a scalar (number density) and vector (energy-momentum density) in special and general relativity. This allows us to introduce the stress-energy tensor which requires a fundamental discussion about its components. We then introduce the conservation equation of energy and momentum in both flat and curved space which plays an important role in describing matter in the universe and thus, motivating the essential meaning of Einstein's equation.

1 Introduction

In this paper, we are building the understanding of the the relation between spacetime curvature and matter energy density. As introduced previously, they are equivalent:

$$\begin{pmatrix} \text{a measure of local} \\ \text{spacetime curvature} \end{pmatrix} = \begin{pmatrix} \text{a measure of} \\ \text{matter energy density} \end{pmatrix} \tag{1}$$

In this paper, we will concentrate on finding the correct measure of energy density that corresponds to the R-H-S of equation 1 in addition to finding the general measure of spacetime curvature for the L-H-S. We begin by discussing densities, starting from its simplest, the number density.

2 Density representation in special and general relativity

$$\text{We assume flat space time : } \left\{ \begin{array}{c} \text{Rectangular coordinates: } (t, x, y, z) \\ \text{Metric: } g_{\alpha\beta} = \eta_{\alpha\beta} = \text{diag}(-1, 1, 1, 1) \end{array} \right.$$

We start our discussion with a simple case: **The number density** (density of a scalar). Consider the following situation where a box containing $\mathcal{N}$ particles is moving with speed V along the x-axis as in 1.

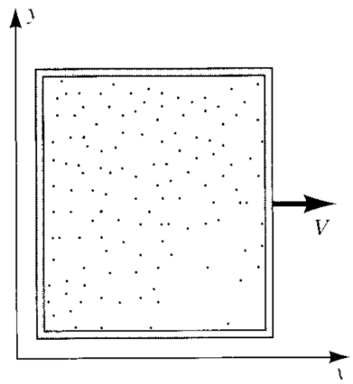

Figure 1: $\mathcal{N}$ particles are trapped in box which moves with speed V along the x-axis (Hartle,2003)

The number density in the rest frame of the box is: $n = \frac{\mathcal{N}}{\mathcal{V}_*}$ where $\mathcal{V}_*$ is the volume of the box in the rest frame.

How would the number density change in the moving frame w.r.t the rest frame?

The number density should become larger since the volume will be smaller due to the Lorentz-contraction of one of its lengths (the length along the x-axis.) by a factor of $(1 - V^2)^{\frac{1}{2}}$. Therefore, the density in the moving frame is :

$$N = \frac{\mathcal{N}}{\mathcal{V}} = \frac{\mathcal{N}}{\mathcal{V}_*(1 - V^2)^{\frac{1}{2}}} = \frac{n}{\sqrt{1 - V^2}} = nu^t \tag{2}$$

Where n is the rest number density and we recall that $u^t = \gamma = (1 - V^2)^{\frac{1}{2}}$ which is the time component of the four-velocity vector of the moving box.

This is the time component of a more generic four vector called the number current four-vector which we define as :

$$\mathbf{N} = n\mathbf{u} \quad ^1 \tag{3}$$

With components: $N^\alpha = (N, \vec{N})$ where:

$$\vec{N} = n\vec{u} = \frac{n\vec{V}}{\sqrt{1 - V^2}}$$

Which is the spatial component of $\mathbf{N}$.
Next, note that if $\mathcal{V}_*$ is taken to be very small, we can define $N(x)$ and $\vec{N}(x)$ at a point x in spacetime. These quantities are constrained by conservation of the number of particles.

[1]The bold notation indicates a four-vector.

To see this, we know that:

$$\begin{pmatrix} \text{the time rate of change of} \\ \text{the number of particles} \\ \text{inside volume V} \end{pmatrix} = - \begin{pmatrix} \text{net rate at which particles} \\ \text{flow out through the surfaces} \\ \text{of the volume-walls of the box} \end{pmatrix}$$

Mathematically,

$$\frac{d}{dt}\int_{\mathcal{V}} N d^3x = -\int_{\partial\mathcal{V}} \vec{N}\cdot d\vec{A} \tag{4}$$

By the divergence theorem:

$$\frac{d}{dt}\int_{\mathcal{V}} N d^3x = -\int_{\mathcal{V}} \nabla\cdot\vec{N}$$

Which yields:

$$\frac{\partial N}{\partial t} = -\nabla\cdot\vec{N} \tag{5}$$

Which can be expressed compactly as:

$$\frac{\partial N^\alpha}{\partial x^\alpha} = 0 \tag{6}$$

Where $N^\alpha = (N, \vec{N})$ and $x^\alpha = (t, x, y, z)$

Generally, the densities of scalar quantities are the time component of a 4-vector whose spatial component is referred to as the current density.

To understand this better, lets look at it geometrically. The density is a function that takes a scalar and outputs an 3-dimensional element. As discussed in previous lectures, a 3-dimensional element can be seen as a three-surface in four-dimensional space whose orientation is specified by a normal 4-vector. A three-volume element is therefore $\mathbf{n}\Delta\mathcal{V}$.
In order to obtain a scalar quantity associated with this three-volume element, we would need a scalar product of $\mathbf{n}\Delta\mathcal{V}$ with a 4-vector current. For example, if $\Delta\mathcal{N}$ is the number of particles in the three-volume $\mathbf{n}\Delta\mathcal{V}$ then:

$$\Delta\mathcal{N} = \mathbf{N}\cdot(\mathbf{n}\Delta\mathcal{V}) = N^\alpha n_\alpha \Delta\mathcal{V} \tag{7}$$

Where $\mathbf{N}$ is the number current density.

Therefore, the spatial density can be thought of as the flux of $\mathbf{N}$ through an element of spacelike three-surface. Whereas the current is a flux in spacelike directions through timelike three surfaces.

3 Densities of energy and momentum

We discussed the density of a scalar (number density), we will now consider the density of vectors, in particular, the energy and momentum vectors since they are the sources of spacetime curvature in the right hand side of Einstein's equation in equation 1.
Similarly to the relation in equation 7, we can associate a four-vector Δp^α with a three volume $n_\alpha \Delta \mathcal{V}$ through $T^{\alpha\beta}$ which has to be a second rank tensor, this is expressed as follows:

$$\Delta p^\alpha = T^{\alpha\beta} n_\beta \Delta \mathcal{V} \tag{8}$$

Where $T^{\alpha\beta}$ is called the energy-momentum-stress tensor or the stress-energy tensor.

To find out the components of $T^{\alpha\beta}$ we consider an inertial frame in flat spacetime that contains a 3-D volume $\Delta\mathcal{V}$. As seen before, this volume is part of a three surface in spacetime defined by:

$$\{ \begin{array}{c} -t = constant \\ n_\alpha = (1,0,0,0) \text{ is the normal to the three surface} \end{array}$$

We can see this by taking a vector with $t = 0$, for example, then $(0, \vec{x}) = x_\alpha$ is a tangent to the three surface. Thus, $\mathbf{n} \times \mathbf{t} = 0$ which yields $n_\alpha = (1,0,0,0)$.
Therefore, we can write equation 8 as $\Delta p^\alpha = T^{\alpha t} \Delta\mathcal{V}$
We define the energy density ϵ and the moment density $\vec{\pi}$ as follows :

$$\{ \begin{array}{l} \epsilon \equiv \Delta p^t / \Delta\mathcal{V} = T^{tt} \\ \vec{\pi} \equiv \Delta p^i / \Delta\mathcal{V} = T^{it} \end{array} \tag{9}$$

Which can be both measured by an observer at rest in the inertial frame.
To illustrate the stress-energy tensor, consider 1 again and suppose that the particles, with mass m, are at rest w.r.t the box. For each particle, we have $E_{particle} = m\gamma$ and $P_{]article} = m\vec{v}\gamma$ with $\gamma = \left(1 - V^2\right)^{\frac{-1}{2}}$. We also know that in order to obtain the energy density, we need to multiply the the number density by the energy, which can be expressed as:

$$\epsilon = N.E_{particle}$$

which according to equation 9 and equation 6, would be equivalent to:

$$\epsilon \equiv T^{tt} = mn\gamma^2 = mnu^t u^t \tag{10}$$

Similarly, the momentum density can be written as:

$$\pi^l \equiv T^{lt} = mn\gamma^2 V^l = mnu^l u^t \tag{11}$$

We can guess the more general expression of the stress-energy tensor to be:

$$T^{\alpha\beta} = mnu^{\alpha}u^{\beta} \equiv \mu u^{\alpha}u^{\beta} \tag{12}$$

Notice that it is symmetric since $T^{\alpha\beta} = T^{\beta\alpha}$. Now, we already explained what $T^{\alpha t}$ means in terms of the energy and momentum density, we should then take a look at the meaning of the $T^{\alpha j}$ components.

For the time components:
Consider a time-like 3-surface and a 3-volume spanned by $\Delta y, \Delta z,$ and Δt. The unit normal to this surface is along x so $n_{\alpha} = (0, 1, 0, 0)$.
Therefore,

$$\Delta p^{\alpha} = T^{\alpha x}\mathcal{V}$$

becomes $\Delta p^{\alpha} = T^{\alpha x}\Delta y\Delta z\Delta t(13)$ and solving for T^{tx}, we obtain:

$$T^{\mathrm{tx}} = \frac{\Delta p^{t}}{\Delta A\Delta t} \tag{14}$$

Which is the flux of energy in the x-direction which is equivalent to the momentum density. To see this, consider the box from figure 1 again, the amount of energy that cross a surface dA in the y and z plane in a time dt is

$$(\text{energy flux})dAdt = (\text{ energy density })VdAdt = (\text{ momentum density })dAdt$$

which implies

$$T^{tx} = T^{xt}$$

The L-H-S is the flux of energy as defined in equation 14 and the R-H-S is the momentum density as defined in equation 11.

For the spatial components: From equation 13, we have $\Delta p^{l} = T^{lx}\Delta A\Delta t$ which implies that:

$$T^{lx} = \frac{\Delta p^{l}/\Delta t}{\Delta A} \tag{15}$$

The rate of change of momentum $\Delta p^{l}/\Delta t$ is just the force and ΔA is the area from the surface whose normal is parallel to the x-axis. Therefore,

$$\Delta F^{i} = T^{lJ}n_{J}\Delta A \tag{16}$$

with T^{lJ} being the i^{th} component of the force per unit area exerted across a surface with normal in direction j.

To summarize all the components that we explained above:

- T^{tt} is the energy density
- T^{it} is the momentum density along i
- T^{ti} is the energy flux along i
- T^{ij} is the stress tensor

This can be expressed more compactly as:

$$T^{\alpha\beta} = \left(\begin{array}{c|c} \text{energy density} & \text{energy flux} \\ \hline \text{momentum} & \text{stress} \\ \text{density} & \text{tensor} \end{array}\right) \tag{17}$$

Again, the stress-energy tensor as a whole is also symmetric, implying $T^{\alpha\beta} = T^{\beta\alpha}$.

4 Conservation of Energy and Momentum

We first consider flat-spacetime. Both the energy and momentum of matter are conserved. Similarly to equation 6 we have in flat spacetime:

$$\frac{\partial T^{\alpha\beta}}{\partial x^{\beta}} = 0 \tag{18}$$

where α is a free index which implied that there are 4 equations, an energy equation and the three components of momentum.

Again, similarly to equation 5, we have for the time component:

$$\frac{\partial T^{tt}}{\partial t} + \vec{\nabla} \cdot T^{it} = 0 \tag{19}$$

Per the definitions of T^{tt} being the energy density and T^{it} being the momentum density(equivalent to the energy flux), equation 19 is equivalent to:

$$\frac{\partial \epsilon}{\partial t} + \vec{\nabla} \cdot \vec{\pi} = 0 \tag{20}$$

which expresses the energy conservation.
The first term, if integrated over a small volume in space, is the rate of change of energy inside the volume while the second term, if integrated the same way, is the flux of energy going outside.

To derive the equation of motion of the fluid, equation 18 implies that the spatial components satisfy:

$$\frac{\partial \pi^{l}}{\partial t} = -\frac{\partial T^{lJ}}{\partial x^{j}} \equiv \phi^{l} \tag{21}$$

Where ϕ is a force density since it is equal to the change in momentum density. To obtain the force acting on a small fixed volume, we can integrate ϕ^l over this volume and use the divergence theorem as follows:

$$F^l = \int_{\mathcal{V}} d^3x\phi^l = -\int_{\mathcal{V}} d^3x \frac{\partial T^{lJ}}{\partial x^J} = -\int_{\partial\mathcal{V}} dAn_J^{(\text{out})} T^{iJ} = \int_{\partial\mathcal{V}} dAn_J^{(\text{in})} T^{lJ} \tag{22}$$

Where $\partial\mathcal{V}$ is the boundary of $\mathcal{V}$ and $n_J^{(\text{out})}$, $n_J^{(\text{in})}$ are the outward and inward normals to $\partial\mathcal{V}$, respectively. By equation 16, we can deduce that the R-H-S is equivalent to $\int_{\partial\mathcal{V}} \Delta F$ which is the sum of the the forces exerted across the boundary surface.

To illustrate, we will discuss an important example for a perfect fluid. Here, the stress tensor is equivalent to the pressure. To briefly motivate this, equation 15 implies that the stress tensor is the force / area which is, for a fluid at rest, is equivalent to the pressure. Since this force is always normal across the surface of the fluid, the i^{th} component is along $i = j$ which implies that T^{ij} is diagonal, leading to

$$T^{ij} = \delta^{ij} p$$

.

Since the time component of the stress energy is the energy density, for a perfect fluid in the rest frame, we can write the following:

$$T^{\alpha\beta} = \text{diag}(\rho, p, p, p) \tag{23}$$

So what happens when a fluid is moving?
Let the four-velocity of the moving fluid be $\mathbf{u}(x)$. We know that $T^{\alpha\beta}$ should depend on $\mathbf{u}(x)$, $\rho(x)$, $p(x)$, and the flat space metric $\eta^{\alpha\beta}$. The most general expression for this dependence would be a linear combination (excluding the possibility of containing derivatives) is :

$$T^{\alpha\beta} = Au^\alpha u^\beta + B\eta^{\alpha\beta} \tag{24}$$

Where A and B can be found by the requirement that equation 24 should reduce to equation 23 in the frame of an observer at rest w.r.t to the fluid. This implies that $u^\alpha = (1, \overrightarrow{0})$ which yields:

$$\{ \begin{array}{c} T^{11} = Au^1u^1 + B\eta^{11} = \rho = A - B \\ T^{22} = Au^2u^2 + B\eta^{22} = p = B \end{array}$$

Which implies that $A = \rho + p$ and $B = p$. Thus, the perfect fluid stress energy is given by:

$$T^{\alpha\beta} = (\rho + p)u^\alpha u^\beta + \eta^{\alpha\beta} p \tag{25}$$

This equation is used to model matter in many situations, for instance, the matter inside neutron stars, gas in galaxies, the cosmic microwave background..etc.

Let us now move on to curved spacetime. This is particularly important since Einstein's equation is clearly intending to describe a curved spacetime.

The perfect fluid stress energy can be easily generalized as follows:

$$T^{\alpha\beta} = (\rho + p)u^{\alpha}u^{\beta} + g^{\alpha\beta}p \tag{26}$$

Which naturally reduces to equation 27 in a local inertial frame.

As for the equation of conservation of energy and momentum (equation 18), it generalizes as follows:

$$\nabla_{\beta}T^{\alpha\beta} = 0 \tag{27}$$

Where ∇_{β} is the covariant derivative (how a second rank tensor is differentiated). Equation 27 is actually not a conservation law, despite the name, since energy is not conserved in the presence of dynamic spacetime curvature, it changes as a response to the change in the curvature. For instance, the cosmic microwave background radiation experiences a decrease in energy and temperature as the universe expands which can be described by equation 27.

5 Conclusion

Starting out from understanding densities in special and general relativity, we built our way to the full meaning of the R-H-S of Einstein's equation, which is contained in the energy tensor of equation 17. We then described the conservation of energy and momentum through equation 18 for flat space and 27 for curved space..
At this stage, we are ready to combine all the knowledge we gained throughout this paper to introduce Einstein's equation, which will be covered in the next lecture with David.

References

Hartle, J. (2003). Gravity.