University of Rochester

Quantum Field Theory II

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Complex Solutions to the Klein-Gordon Equation

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1 Introduction

At this point in the course, the Klein-Gordon equation had been studied fairly thoroughly. We have derived the Lagrangian and Hamiltonian densities that would result in the Klein-Gordon equation and defined the field operator in terms of the creation and annihilation operators. However, when initially developing the tools needed to derive this information, we made a key assumption in that we stipulated that the field operator must be a real valued function. While this allowed us to develop many important results, ultimately it ignores many solutions to the equation. Therefore, it is my goal to show how the earlier results of the purely real solutions could be applied when the field operator is complex.

2 Derivation of the Lagrangian and Hamiltonian

2.1 Available Equations

As before, we have the original Klein-Gordon equation

$$(\partial_\mu \partial^\mu + m^2)\phi(x) = 0. \tag{1}$$

However, since we also know that $\phi \neq \phi^\dagger$, we also have

$$(\partial_\mu \partial^\mu + m^2)\phi^\dagger(x) = 0. \tag{2}$$

We can also write ϕ and $\phi^\dagger$ in terms of purely real functions by taking one real function to act as the real part of ϕ and another to be the imaginary part. Therefore, we have

$$\phi(x) = \frac{1}{\sqrt{2}}\left(\phi_1(x) + i\phi_2(x)\right) \tag{3}$$

$$\phi^\dagger(x) = \frac{1}{\sqrt{2}}\left(\phi_1(x) - i\phi_2(x)\right). \tag{4}$$

We can also invert these relations in order to find ϕ_1 and ϕ_2 in terms of ϕ and $\phi^\dagger$

$$\phi_1(x) = \frac{1}{\sqrt{2}}(\phi(x) + \phi^\dagger(x)) \tag{5}$$

$$\phi_2(x) = \frac{-i}{\sqrt{2}}(\phi(x) - \phi^\dagger(x)). \tag{6}$$

By substituting (3) and (4) into (1) and (2), we can easily see that we have

$$(\partial_\mu \partial^\mu + m^2)\phi_1(x) = 0 \tag{7}$$

$$(\partial_\mu \partial^\mu + m^2)\phi_2(x) = 0. \tag{8}$$

Therefore, a system that can be described by a complex solution to the Klein-Gordon equation also be described by a system of two independent particles with equal mass that have real solutions to the Klein-Klein-Gordon equation.

2.2 Lagrangian Density

We have two potential ways to describe this complex system, with ϕ and $\phi^\dagger$ or with ϕ_1 and ϕ_2. First, we look at the Lagrangian density using ϕ_1 and ϕ_2. Using the earlier definition of the Lagrangian density for real field operators, we have

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{m^2}{2}\phi_1^2 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{m^2}{2}\phi_2^2 \\ &= \frac{1}{2}(\partial_\mu\phi_1\partial^\mu\phi_1 + \partial_\mu\phi_2\partial^\mu\phi_2) - \frac{m^2}{2}(\phi_1^2 + \phi_2^2).\end{aligned} \tag{9}$$

However, by factoring the terms in this Lagrangian density, we can rewrite it in terms of ϕ and $\phi^\dagger$

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\partial_\mu(\phi_1 - i\phi_2)\partial^\mu(\phi_1 + i\phi_2) - \frac{m^2}{2}(\phi_1 - i\phi_2)(\phi_1 + i\phi_2) \\ &= \partial_\mu\phi^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi. \end{aligned} \tag{10}$$

We note that even though ϕ and $\phi^\dagger$ are clearly not Hermitian, the Lagrangian density which describes this is in fact Hermitian.

2.3 Hamiltonian Density

2.3.1 ϕ_1 and ϕ_2

Now that we have the Lagrangian density, we can determine what the Hamiltonian density for this system will be. We will first define the Hamiltonian density in terms of ϕ_1 and ϕ_2. First, we define the momentum operators

$$\Pi_i(x) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}_i} = \dot{\phi}_i(x), \qquad i = 1, 2 \tag{11}$$

We also define the commutation relations for these momentum operators as well as the ϕ_i. Using the results for real solutions to the Klein-Gordon equation and the fact these ϕ_i are independent of each other, we determine that

$$[\phi_i(x), \phi_j(y)] = [\Pi_i(x), \Pi_j(y)] = 0 \tag{12}$$

$$[\phi_i(x), \Pi_j(y)] = i\delta_{ij}\delta^3(x - y) \tag{13}$$

Then, we use the standard definition of the Hamiltonian density for a system with two variables

$$\begin{aligned}\mathcal{H} &= \sum_{i=1}^{2}\Pi_i\dot{\phi}_i - \mathcal{L} \\ &= \sum_{i=1}^{2}\Pi_i^2 - \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{m^2}{2}\phi_1^2 - \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 + \frac{m^2}{2}\phi_2^2 \\ &= \sum_{i=1}^{2}\Pi_i^2 - \frac{1}{2}\dot{\phi}_1^{\,2} + \frac{1}{2}\nabla\phi_1\cdot\nabla\phi_1 + \frac{m^2}{2}\phi_1^2 - \frac{1}{2}\phi_2^2 + \frac{1}{2}\dot{\phi}_2^{\,2} + \frac{1}{2}\nabla\phi_2\cdot\nabla\phi_2 + \frac{m^2}{2}\phi_2^2 \\ &= \sum_{i=1}^{2}\left(\frac{1}{2}\Pi_i^2 + \frac{1}{2}\nabla\phi_i\cdot\nabla\phi_i + \frac{m^2}{2}\phi_i^2\right) \end{aligned} \tag{14}$$

Therefore, we can write the Hamiltonian density of the system as the sum of the two Hamiltonian densities for ϕ_1 and ϕ_2.

2.3.2 ϕ and $\phi^\dagger$

As before, we define the momentum operators as follows:

$$\begin{aligned}\Pi(x) &= \frac{\partial\mathcal{L}}{\partial\dot{\phi}^\dagger(x)} = \dot{\phi} \\ &= \frac{1}{\sqrt{2}}\left(\dot{\phi}_1(x) + i\dot{\phi}_2(x)\right) \\ &= \frac{1}{\sqrt{2}}\left(\Pi_1(x) + i\Pi_2(x)\right) \end{aligned} \tag{15}$$

We can find the complex conjugate of the momentum operator in a similar fashion

$$\begin{aligned}\Pi^\dagger(x) &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)} = \dot{\phi}^\dagger \\ &= \frac{1}{\sqrt{2}}\left(\Pi_1(x) - i\Pi_2(x)\right)\end{aligned} \tag{16}$$

As before, we can determine the commutation relations of the system by noting that the ϕ and $\phi^\dagger$ are independent variables.

$$[\phi(x), \phi(y)] = [\phi(x), \phi^\dagger(y)] = [\phi^\dagger(x), \phi^\dagger(y)] = 0 \tag{17}$$

$$[\Pi(x), \Pi(y)] = [\Pi(x), \Pi^\dagger(y)] = [\Pi^\dagger(x), \Pi^\dagger(y)] = 0 \tag{18}$$

$$[\phi(x), \Pi^\dagger(y)] = [\phi^\dagger(y), \Pi(y)] = i\delta^3(x-y) \tag{19}$$

The Hamiltonian density therefore can be written as

$$\begin{aligned}\mathcal{H} &= \Pi\dot{\phi}^\dagger + \Pi^\dagger\dot{\phi} - \mathcal{L} \\ &= \Pi^\dagger\Pi + \Pi^\dagger\Pi - \dot{\phi}^\dagger\dot{\phi} + \nabla\phi^\dagger\cdot\nabla\phi + m^2\phi^\dagger\phi \\ &= \Pi^\dagger\Pi + \nabla\phi^\dagger\cdot\nabla\phi + m^2\phi^\dagger\phi\end{aligned} \tag{20}$$

Like the Lagrangian density, this Hamiltonian density is Hermitian, even though ϕ and $\phi^\dagger$ are not.

3 Quantizing the Field

We have already begun to see the correlations between the complex solutions and real solutions to the Klein-Gordon equation. When quantizing this field, we can therefore apply the results we obtained before to the real and imaginary parts of ϕ. We can therefore describe the ϕ_1 and ϕ_2 in terms of the creation and annihilation operators in the same way as before

$$\phi_i(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2k^0}}\left(e^{-ik\cdot x}a_i(\mathbf{k}) + e^{ik\cdot x}a_i^\dagger(\mathbf{k})\right), \qquad i = 1, 2 \tag{21}$$

These creation and annihilation are defined in the same way as for a single real solution of the Klein-Gordon equation. Therefore, the commutation relations can be written as

$$[a_i(\mathbf{k}), a_j(\mathbf{k}')] = [a_i^\dagger(\mathbf{k}), a_j^\dagger(\mathbf{k}')] = 0 \tag{22}$$

$$[a_i(\mathbf{k}), a_j^\dagger(\mathbf{k}')] = \delta_{ij}\delta^3(k-k') \tag{23}$$

By substituting (21) into (3) and (4), we can also determine the definitions for ϕ and $\phi^\dagger$ in terms of creation and annihilation operators.

$$\begin{aligned}\phi(x) &= \frac{1}{\sqrt{2}}\left(\phi_1(x) + i\phi_2(x)\right) \\ &= \int \frac{d^3k}{\sqrt{(2\pi)^3 2k^0}}\left[e^{-ik\cdot x}\left(a_1(\mathbf{k}) + ia_2(\mathbf{k})\right) + e^{ik\cdot x}\left(a_1^\dagger(\mathbf{k}) + ia_2^\dagger(\mathbf{k})\right)\right]\end{aligned} \tag{24}$$

In order to simplify the notation, we can define additional creation and annihilation operators $a(\mathbf{k})$ and $b(\mathbf{k})$.

$$a(\mathbf{k}) = \frac{1}{\sqrt{2}}(a_1(\mathbf{k}) + ia_2(\mathbf{k})) \tag{25}$$

$$b(\mathbf{k}) = \frac{1}{\sqrt{2}}(a_1(\mathbf{k}) - ia_2(\mathbf{k})) \tag{26}$$

Then, we can easily define ϕ and $\phi^\dagger$ as

$$\phi(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2k^0}} \left(e^{-ik\cdot x} a(\mathbf{k}) + e^{ik\cdot x} b^\dagger(\mathbf{k})\right) \tag{27}$$

$$\phi^\dagger(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2k^0}} \left(e^{-ik\cdot x} b(\mathbf{k}) + e^{ik\cdot x} a^\dagger(\mathbf{k})\right) \tag{28}$$

We recognize that $a(\mathbf{k})$ and $b(\mathbf{k})$ act as the annihilation operators for ϕ and $\phi^\dagger$, respectively. The commutation relations for these operators are as expected, with all permutations of $a(\mathbf{k})$, $a^\dagger(\mathbf{k})$, $b(\mathbf{k})$, and $b^\dagger(\mathbf{k})$ commuting, except for the following relation:

$$[a(\mathbf{k}), a^\dagger(\mathbf{k}')] = [b(\mathbf{k}), b^\dagger(\mathbf{k}')] = \delta^3(k - k') \tag{29}$$

Now that we have definitions of the field operators in terms of the annihilation and creations operators as well as the Hamiltonian density for this system, we can define the Hamiltonian of the system. By substituting (27) and (28) into (20) and taking the appropriate derivatives, we can see that

$$\begin{aligned} H &= \int d^3x \mathcal{H} \\ &= \int d^3k \frac{E_k}{2} \left(a^\dagger(\mathbf{k})a(\mathbf{k}) + a(\mathbf{k})a^\dagger(\mathbf{k}) + b^\dagger(\mathbf{k})b(\mathbf{k}) + b(\mathbf{k})b^\dagger(\mathbf{k})\right). \end{aligned} \tag{30}$$

We define the normal ordering of an operator $: H :$ as the ordering of the creation and annihilation operators used to define that operator such that the creation operator is to the left of the annihilation operator. Then,

$$: H := \int d^3k\, E_k \left(a^\dagger(\mathbf{k})a(\mathbf{k}) + b^\dagger(\mathbf{k})b(\mathbf{k})\right). \tag{31}$$

We go on to define the vacuum state as we did in the case of the real field operator

$$a(\mathbf{k})|0\rangle = b(\mathbf{k})|0\rangle = \langle 0|a^\dagger(\mathbf{k}) = \langle 0|b^\dagger(\mathbf{k}) = 0 \tag{32}$$

However, since we have two creation operators, we have two different states with the same energy. That is, we have

$$a^\dagger(\mathbf{k})|0\rangle = |k\rangle \tag{33}$$

$$b^\dagger(\mathbf{k})|0\rangle = |\tilde{k}\rangle \tag{34}$$

$$\tag{35}$$

This is not entirely unexpected, as when we originally defined our ϕ in terms of ϕ_1 and ϕ_2, we noted that the system could be described by two Klein-Gordon equations for particles with equal mass, and therefore equal energy. In order to gain a better understanding of this degeneracy, we construct the charge operator for this system.

4 The Charge Operator

4.1 Current

In order to construct the charge operator, we must first define the current, J^μ. To find the this current, we use a definition similar to the current of a real valued field. However, an additional term is added to account for $\phi^\dagger$.

$$J^\mu = \delta\phi(x)\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} + \delta\phi^\dagger(x)\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^\dagger)} - K^\mu, \tag{36}$$

where K^μ is defined by

$$\mathcal{L}\left(\phi'(x), \partial_\mu\phi'(x)\right) - \mathcal{L}\left(\phi(x), \partial_\mu\phi(x)\right) = \partial_\mu K^\mu. \tag{37}$$

In order to define J^μ, we define the transformations to ϕ' and $\phi'^\dagger$ as $\phi(x) \to \phi'(x) = e^{-i\theta}\phi(x)$ and $\phi(x)^\dagger \to \phi'^\dagger(x) = \phi(x)e^{i\theta}$, where θ is parameter independent of the spacetime coordinates. Then, taking the variation in the Lagrangian, we have

$$\begin{aligned}
\mathcal{L}\left(\phi'(x), \partial_\mu\phi'(x)\right) - \mathcal{L}\left(\phi(x), \partial_\mu\phi(x)\right) &= \partial_\mu\phi'^\dagger\partial^\mu\phi' - m^2\phi'^\dagger\phi' - \partial_\mu\phi^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi \\
&= \partial_\mu\phi^\dagger e^{i\theta}e^{-i\theta}\partial^\mu\phi - m^2\phi^\dagger e^{i\theta}e^{-i\theta}\phi - \partial_\mu\phi^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi \\
&= 0.
\end{aligned}$$

This allows us to choose K^μ to be 0. To determine the variation in ϕ and $\phi^\dagger$, we take an infinitesimal transformation $\theta = \epsilon$. Then we have

$$\delta\phi(x) = \phi'(x) - \phi(x) = e^{-i\epsilon}\phi(x) - \phi(x) = -i\epsilon\phi(x) \tag{38}$$

$$\delta\phi^\dagger(x) = i\epsilon\phi^\dagger(x) \tag{39}$$

With this, $K^\mu = 0$ and the definition of the Lagrangian in (10), we can finally determine the current. We have

$$\begin{aligned}
J^\mu &= -i\epsilon\phi(x)\partial^\mu\phi^\dagger(x) + i\epsilon\phi^\dagger(x)\partial^\mu\phi(x) \\
&= i\epsilon\left(\phi^\dagger(x)\partial^\mu\phi(x) - \phi(x)\partial^\mu\phi^\dagger(x)\right) \\
&= i\epsilon\phi^\dagger(x)\overleftrightarrow{\partial^\mu}\phi(x).
\end{aligned} \tag{40}$$

We can then define the parameter free current as $J^\mu = i\phi^\dagger\overleftrightarrow{\partial^\mu}\phi(x)$. Then, we can determine the charge operator Q from the definition

$$\begin{aligned}
Q &= \int d^3x J^0(x) = i\int d^3x \phi^\dagger\overleftrightarrow{\partial^\mu}\phi(x) \\
&= i\int d^3x \left(\phi^\dagger\dot{\phi}(x) - \dot{\phi}^\dagger(x)\phi(x)\right) \\
&= \int d^3k \left(a^\dagger(\mathbf{k})a(\mathbf{k}) - b(\mathbf{k})b^\dagger(\mathbf{k})\right) \\
\Longrightarrow :Q: &= \int d^3k \left(a^\dagger(\mathbf{k})a(\mathbf{k}) - b^\dagger(\mathbf{k})b(\mathbf{k})\right)
\end{aligned} \tag{41}$$

4.2 Using the Charge Operator

We can now use the charge operator in an attempt to gain a better understanding of the system. By inspection, we can see that $Q|0\rangle = 0$, so the vacuum contains no charge. However, when we apply this charge operator to $|k\rangle$, we obtain

$$\begin{aligned}
Q|k\rangle &= \int d^3k' \left(a^\dagger(\mathbf{k}')a(\mathbf{k}') - b^\dagger(\mathbf{k}')b(\mathbf{k}')\right)a^\dagger(\mathbf{k})|0\rangle \\
&= \int d^3k' a^\dagger(\mathbf{k}')a(\mathbf{k}')a^\dagger(\mathbf{k})|0\rangle \\
&= \int d^3k' a^\dagger(\mathbf{k}')\left([a(\mathbf{k}'), a^\dagger(\mathbf{k})] + a^\dagger(\mathbf{k}), a(\mathbf{k}')\right)|0\rangle \\
&= \int d^3k' a^\dagger(\mathbf{k}')\delta^3(\mathbf{k}' - \mathbf{k})|0\rangle = a^\dagger(\mathbf{k})|0\rangle = |k\rangle.
\end{aligned} \tag{42}$$

Therefore, the $|k\rangle$ is an eigenvector of the charge operator with eigenvalue. A similar calculation can be done on $|\tilde{k}\rangle$ to determine that

$$Q|\tilde{k}\rangle = -|\tilde{k}\rangle. \tag{43}$$

This shows that $|\tilde{k}\rangle$ is also an eigenvector of Q with eigenvalue -1. This indicates that the two particles we described earlier have opposite charge and that may be recognized as a particle and its respective antiparticle. This allows to do away with the Dirac's hole theory and assign antiparticles positive energy. This also leads us to the conclusion that the Klein-Gordon equation with only field operators describes neutral particles, while allowing complex solutions describes particles with electric charge.

5 The Feynman Green's Function

Now that we have have determined many of the properties of the complex field operator, we can use it to define the Feynman Green's function. We begin by noting that

$$\begin{aligned}
\langle 0|\phi(x)\phi(y)|0\rangle &= \int\int \frac{d^3k d^3k'}{2(2\pi)^3\sqrt{k^0 k'^0}} \langle 0| \left(e^{-ik\cdot x}a(\mathbf{k}) + e^{ik\cdot x}b^\dagger(\mathbf{k})\right) \cdot \left(e^{-ik'\cdot y}a(\mathbf{k'}) + e^{ik'\cdot y}b^\dagger(\mathbf{k'})\right)|0\rangle \\
&= \int\int \frac{d^3k d^3k'}{2(2\pi)^3\sqrt{k^0 k'^0}} \left(e^{-ik\cdot x}\langle 0|a(\mathbf{k}) + e^{ik\cdot x}\langle 0|b^\dagger(\mathbf{k})\right) \cdot \left(e^{-ik'\cdot y}a(\mathbf{k'}))|0\rangle + e^{ik'\cdot y}b^\dagger(\mathbf{k'}))|0\rangle\right) \\
&= \int\int \frac{d^3k d^3k'}{2(2\pi)^3\sqrt{k^0 k'^0}} e^{-ik\cdot x}e^{ik'\cdot y}\langle 0|a(\mathbf{k})b^\dagger(\mathbf{k'})|0\rangle \\
&= 0,
\end{aligned} \tag{44}$$

where in the last step we have used the fact that $a(\mathbf{k})$ and $b^\dagger(\mathbf{k'})$ commute. Using a similar argument, we can also see that

$$\langle 0|\phi^\dagger(x)\phi^\dagger(y)|0\rangle = 0 \tag{45}$$

However, the result is not as trivial when taking a combination of ϕ and $\phi^\dagger$. In this case, we have

$$\begin{aligned}
\langle 0|\phi(x)\phi^\dagger(y)|0\rangle &= \int\int \frac{d^3k d^3k'}{2(2\pi)^3\sqrt{k^0 k'^0}} \langle 0| \left(e^{-ik\cdot x}a(\mathbf{k}) + e^{ik\cdot x}b^\dagger(\mathbf{k})\right) \cdot \left(e^{-ik'\cdot y}b(\mathbf{k'}) + e^{ik'\cdot y}a^\dagger(\mathbf{k'})\right)|0\rangle \\
&= \int\int \frac{d^3k d^3k'}{2(2\pi)^3\sqrt{k^0 k'^0}} \left(e^{-ik\cdot x}\langle 0|a(\mathbf{k}) + e^{ik\cdot x}\langle 0|b^\dagger(\mathbf{k})\right) \cdot \left(e^{-ik'\cdot y}b(\mathbf{k'}))|0\rangle + e^{ik'\cdot y}a^\dagger(\mathbf{k'}))|0\rangle\right) \\
&= \int\int \frac{d^3k d^3k'}{2(2\pi)^3\sqrt{k^0 k'^0}} e^{-ik\cdot x}e^{ik'\cdot y}\langle 0|a(\mathbf{k})a^\dagger(\mathbf{k'})|0\rangle \\
&= \int\int \frac{d^3k d^3k'}{2(2\pi)^3\sqrt{k^0 k'^0}} e^{-ik\cdot x}e^{ik'\cdot y}\langle 0|[a(\mathbf{k}), a^\dagger(\mathbf{k'})] + a^\dagger(\mathbf{k'})a(\mathbf{k})|0\rangle \\
&= \int\int \frac{d^3k d^3k'}{2(2\pi)^3\sqrt{k^0 k'^0}} e^{-ik\cdot x}e^{ik'\cdot y}\langle 0|\delta^3(k-k')|0\rangle \\
&= \int \frac{d^3k}{(2\pi^3)2k^0} e^{-ik(x-y)} = -iG^{(+)}(x-y).
\end{aligned} \tag{46}$$

Therefore, we can define the positive Feynman Green's function. To obtain the negative Feynman Green's function, we perform a similar process

$$
\begin{aligned}
\langle 0|\phi^\dagger(y)\phi(x)|0\rangle &= \int\int \frac{d^3k d^3k'}{2(2\pi)^3\sqrt{k^0k'^0}} \langle 0|\left(e^{-ik\cdot y}b(\mathbf{k}) + e^{ik\cdot y}a^\dagger(\mathbf{k})\right)\cdot\left(e^{-ik'\cdot x}a(\mathbf{k}') + e^{ik'\cdot x}b^\dagger(\mathbf{k}')\right)|0\rangle \\
&= \int\int \frac{d^3k d^3k'}{2(2\pi)^3\sqrt{k^0k'^0}} \left(e^{-ik\cdot y}\langle 0|b(\mathbf{k}) + e^{ik\cdot y}\langle 0|a^\dagger(\mathbf{k})\right)\cdot\left(e^{-ik'\cdot x}a(\mathbf{k}'))|0\rangle + e^{ik'\cdot x}b^\dagger(\mathbf{k}'))|0\rangle\right) \\
&= \int\int \frac{d^3k d^3k'}{2(2\pi)^3\sqrt{k^0k'^0}} e^{-ik\cdot y}e^{ik'\cdot x}\langle 0|b(\mathbf{k})b^\dagger(\mathbf{k}')|0\rangle \\
&= \int\int \frac{d^3k d^3k'}{2(2\pi)^3\sqrt{k^0k'^0}} e^{-ik\cdot y}e^{ik'\cdot x}\langle 0|[b(\mathbf{k}), b^\dagger(\mathbf{k}')] + b^\dagger(\mathbf{k}')b(\mathbf{k})|0\rangle \\
&= \int\int \frac{d^3k d^3k'}{2(2\pi)^3\sqrt{k^0k'^0}} e^{-ik\cdot y}e^{ik'\cdot x}\langle 0|\delta^3(k-k')|0\rangle \\
&= \int \frac{d^3k}{(2\pi^3)2k^0} e^{ik(x-y)} = iG^{(-)}(x-y). \qquad (47)
\end{aligned}
$$

To get the total Green function, we define the time ordering operator as follows

$$
T(A(t_1)B(t_2)) = \theta(t_1 - t_2)A(t_1)B(t_2) + \theta(t_2 - t_1)B(t_2)A(t_1). \qquad (48)
$$

Then, we apply this time ordering operator to obtain the full Green's function

$$
\begin{aligned}
\langle 0|T(\phi(x)\phi^\dagger(y)|0\rangle &= \theta(x^0 - y^0)\langle 0|\phi(x)\phi^\dagger(y)|0\rangle + \theta(y^0 - x^0)\langle 0|\phi^\dagger(y)\phi(x)|0\rangle \\
&= -i\theta(x^0 - y^0)G^{(+)}(x-y) + i\theta(y^0 - x^0)G^{(-)}(x-y). \\
&= iG_F(x-y) \qquad (49)
\end{aligned}
$$

Therefore, we see that the Feynman Green's function can be defined in terms of the complex field operator and the vacuum state.

6 Introducing Interactions

The complex field Klein-Gordon equation also have interactions, much like in the real case. We begin by introducing a quartic interaction to the Lagrangian density, as we did in the case of the real field operator. Then, the Lagrangian becomes

$$
\mathcal{L} = \partial_\mu\phi^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi - \frac{\lambda}{4}(\phi^\dagger\phi)^2, \quad \lambda > 0. \qquad (50)
$$

We also wish to define this Lagrangian density in terms of the real field operators, ϕ_1 and ϕ_2. Substituting (5) and (6) into (50), we find that the Lagrangian density is

$$
\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_1\partial^\mu\phi_1 + \partial_\mu\phi_2\partial^\mu\phi_2) - \frac{m^2}{2}(\phi_1^2 + \phi_2^2) - \frac{\lambda}{16}(\phi_1^2 + \phi_2^2)^2 \qquad (51)
$$

We then define the Hamiltonian of this system as

$$
H = \int d^3x\, \mathcal{H} = \int d^3x \left(\Pi^\dagger\Pi + \nabla\phi^\dagger\cdot\nabla\phi + m^2\phi^\dagger\phi + \frac{\lambda}{4}(\phi^\dagger\phi)^2\right) \qquad (52)
$$

We attempt to find the potential energy of this system. We note that the minimum of the Hamiltonian corresponds to the minimum of the potential energy, since any kinetic energy terms will be positive. Therefore we have,

$$V = m^2\phi^\dagger\phi + \frac{\lambda}{4}(\phi^\dagger\phi)^2 \tag{53}$$

By inspection, we can see that the minimum of the potential is 0, since it is composed of purely nonnegative terms. By considering the concept of a classical field (which we label as ϕ_c), we say that the minimum occurs at $\phi_c = \phi_c^\dagger = \phi_{1c} = \phi_{2c} = 0$. From classical mechanics, we know that perturbations about a minimum in the potential are stable. In this context, this means that perturbations about the vacuum state are stable.

Now we consider a slightly more interesting. Take the mass $m \to im$. The Lagrangian is now

$$\mathcal{L} = \partial_\mu\phi^\dagger\partial^\mu\phi + m^2\phi^\dagger\phi - \frac{\lambda}{4}(\phi^\dagger\phi)^2 \tag{54}$$

$$= \frac{1}{2}(\partial_\mu\phi_1\partial^\mu\phi_1 + \partial_\mu\phi_2\partial^\mu\phi_2) + \frac{m^2}{2}(\phi_1^2+\phi_2^2) - \frac{\lambda}{16}(\phi_1^2+\phi_2^2)^2 \tag{55}$$

First we consider the noninteracting case, when $\lambda = 0$. Looking at the Klein-Gordon equation, we now have

$$\begin{aligned}
&\left(\partial_\mu\partial^\mu - m^2\right)\phi(x) = 0 \\
&\implies k^2 = -m^2 \\
&\implies k^0 = \sqrt{\mathbf{k}^2 - m^2} \\
&\implies \frac{\partial k^0}{\partial|\mathbf{k}|} = \frac{|\mathbf{k}|}{\sqrt{\mathbf{k}^2 - m^2}} > 1
\end{aligned}$$

This means that the particle must be traveling faster than the speed of light. These particles are called tachyons, and are not included in most theories, as they can violate the normal rules of causality we have set forth. However, we can still consider a potential with $\lambda > 0$. In this case,

$$V = -m^2\phi^\dagger\phi + \frac{\lambda}{4}(\phi^\dagger\phi)^2 \qquad = -\frac{m^2}{2}(\phi_1^2+\phi_2^2) + \frac{\lambda}{16}(\phi_1^2+\phi_2^2)^2 \tag{56}$$

To find the minimum of this potential, we find the 0's of its first derivative.

$$\begin{aligned}
\frac{\partial V}{\partial\phi_1} &= -m^2\phi_1 + \frac{\lambda}{4}\phi_1(\phi_1^2+\phi_2^2) \\
&= \phi_1\left(-m^2 + \frac{\lambda}{4}(\phi_1^2+\phi_2^2)\right) = 0 \qquad (57) \\
\frac{\partial V}{\partial\phi_2} &= -m^2\phi_2 + \frac{\lambda}{4}\phi_2(\phi_1^2+\phi_2^2) \\
&= \phi_2\left(-m^2 + \frac{\lambda}{4}(\phi_1^2+\phi_2^2)\right) = 0 \qquad (58)
\end{aligned}$$

Either $\phi_{1c} = \phi_{2c} = 0$ or $\phi_{1c}^2 + \phi_{2c}^2 = \frac{4m^2}{\lambda}$. In the latter case, we choose $\phi_{1c} = \frac{\pm 2m}{\sqrt{\lambda}}$ and $\phi_{2c} = 0$. To determine whether these saddle points are minima or maxima, we look at the second derivatives of the potential when evaluated at this points. For $\phi_{1c} = \phi_{2c} = 0$, we have

$$\begin{aligned}
\frac{\partial^2 V}{\partial\phi_i^2} &= -m^2 \qquad\qquad i = 1,2 \\
\frac{\partial^2 V}{\partial\phi_1\partial\phi_2} &= \frac{\partial^2 V}{\partial\phi_2\partial\phi_1} = 0. \qquad (59)
\end{aligned}$$

For $\phi_{1c} = \frac{\pm 2m}{\sqrt{\lambda}}$ and $\phi_{2c} = 0$, we instead get

$$\frac{\partial^2 V}{\partial \phi_1^2} = -m^2 + \frac{3\lambda}{4}\left(\frac{4m^2}{\lambda}\right) = 2m^2$$
$$\frac{\partial^2 V}{\partial \phi_2^2} = -m^2 + \frac{\lambda}{4}\left(\frac{4m^2}{\lambda}\right) = 0$$
$$\frac{\partial^2 V}{\partial \phi_1 \partial \phi_2} = 0 \tag{60}$$

Since the second derivative is negative when $\phi_{1c} = \phi_{2c} = 0$ and positive when $\phi_{1c} = \frac{\pm 2m}{\sqrt{\lambda}}$ and $\phi_{2c} = 0$, the first point corresponds to a local maximum, and the second to a local minimum. This can be seen more clearly in the figure 1. This figure shows the potential for this system, often called the Mexican hat potential.

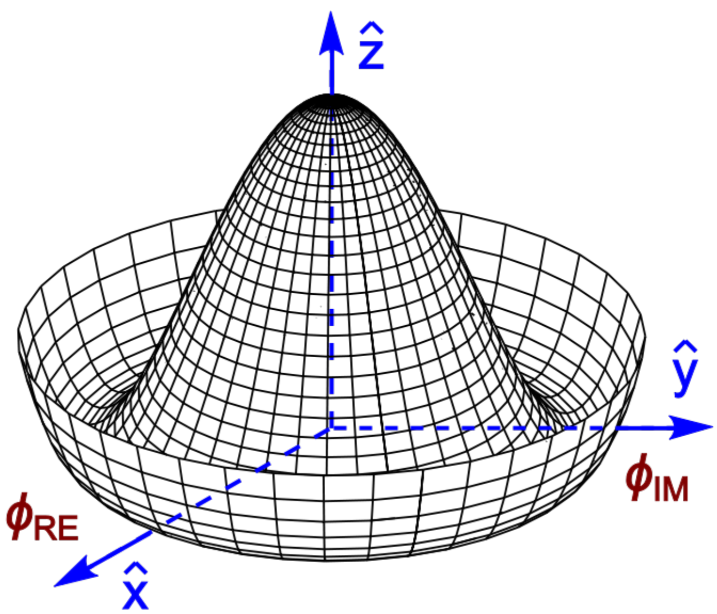

Figure 1: The Mexican hat potential which represents the complex Klein-Gordon equation [2]

7 Conclusion

We can now see that the complex solutions to the Klein-Gordon equation provide a wealth of information that cannot be found simply from the real solutions. The complex solutions allow for an interpretation of anti-particles that does not require negative energies, making it much more attractive than Dirac's hole theory. These solutions also allow the Feynman Green's function to defined in terms of the vacuum state. Lastly, the potential associated with the complex fields is one that leads to spontaneous symmetry breaking, the details of which are unfortunately outside the scope of this paper.

References

[1] Das A. 2008. *Lectures on Quantum Field Theory*, World Scientific, 2008, First Edition, pp. 257-75.

[2] Miller, Rupert `https://upload.wikimedia.org/wikipedia/commons/8/83/Mexican_hat_potential_polar_with_details.svg`