

CALCULATING TRANSITION AMPLITUDES FROM FEYNMAN DIAGRAMS

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1. INTRODUCTION

When one thinks of quantum field theory, one's mind is undoubtedly drawn to Feynman diagrams. The naïve view these diagrams as merely a concise and standardized way of describing collisions and processes. Feynman diagrams, however, encode a great deal of information, for each one corresponds exactly to the probability amplitude of the transition it describes. In other words, it is easily possible to write and calculate the transition amplitude of a given process using only the process' Feynman diagram. Such monumental utility arises thanks to the efforts of Dyson, Wick, and, of course, Feynman.

In this short paper, we will briefly provide some background information on concepts required to understand Feynman diagrams. In particular, we will give an overview of Feynman diagram construction from terms in the S -matrix as expanded by Wick's theorem. We will then calculate a simple transition amplitude by hand. Afterward, we will present the rules that can be used to derive the transition amplitude much more simply from Feynman diagrams.

Throughout this paper, we will refer to equations in the book by Mandl and Shaw and frequently use its notation. Where convenient or deemed sufficiently important, certain equations will be rewritten here. This paper comprises a direct adaptation from the lecture notes I used for Chapter 7, and is in most cases verbatim. Hence the paper assumes a working familiarity with concepts of previous chapters and should not be considered comprehensive.

2. CONSTRUCTING FEYNMAN DIAGRAMS FROM THE S -MATRIX EXPANSION

For future reference, Table 1 succinctly shows the creation and absorption operators of the particles we will be dealing with in QED.

	absorption	creation
e^-	ψ^+	$\bar{\psi}^-$
e^+	$\bar{\psi}^+$	ψ^-
γ	A^+	A^-

TABLE 1. Absorption and creation operators for electrons, positrons, and photons.

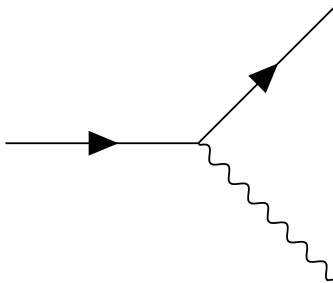


FIGURE 1. The basic Feynman diagram for electron scattering with photon emission.

Recall that the S-matrix expansion is given by

$$(2.1) \quad S = \sum_{n=0}^{\infty} S^{(n)} = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int \cdots \int d^4x_1 \cdots d^4x_n T\{\mathcal{H}_I(x_1) \cdots \mathcal{H}_I(x_n)\},$$

integrated over all spacetime, where the interaction Hamiltonian density is

$$(2.2) \quad \begin{aligned} \mathcal{H}_I(x) &= -eN\{\bar{\psi}(x)\mathcal{A}(x)\psi(x)\} \\ &= -eN\{(\bar{\psi}^+ + \bar{\psi}^-)(\mathcal{A}^+ + \mathcal{A}^-)(\psi^+ + \psi^-)\}_x. \end{aligned}$$

The interaction Hamiltonian density can therefore be split into eight different terms. Each term gives rise to one of eight basic processes, which can be drawn as Feynman diagrams. Figure 1 shows the Feynman diagram for electron scattering with photon emission. Note that $S^{(1)}$ integrates over exactly these terms, and hence corresponds to these basic processes. Since we require that $k^2 = 0$ for photons and $p^2 = m^2$ for fermions, all eight of these basic transitions are unphysical, and so we will find that $\langle f|S^{(1)}|i\rangle = 0$ for any final state vector $|f\rangle$ and initial state vector $|i\rangle$ that these basic processes describe.

In order to get a real, physical process, we must take S to at least its second-order term $S^{(2)}$. Using Wick's theorem, and following the notation in the book, we can write $S^{(2)}$ as

$$(2.3) \quad S^{(2)} = \sum_{i=A}^F S_i^{(2)},$$

where

$$(2.4) \quad S_A^{(2)} = -\frac{e^2}{2!} \int d^4x_1 d^4x_2 N[(\bar{\psi} \underline{A} \psi)_{x_1} (\bar{\psi} \underline{A} \psi)_{x_2}]$$

$$(2.5) \quad S_B^{(2)} = -\frac{e^2}{2!} \int d^4x_1 d^4x_2 \{N[(\bar{\psi} \underline{A} \psi)_{x_1} (\bar{\psi} \underline{A} \psi)_{x_2}] + N[(\bar{\psi} \underline{A} \psi)_{x_1} (\bar{\psi} \underline{A} \psi)_{x_2}]\}$$

$$(2.6) \quad S_C^{(2)} = -\frac{e^2}{2!} \int d^4x_1 d^4x_2 N[(\bar{\psi} \gamma^\alpha \underline{A}_\alpha \psi)_{x_1} (\bar{\psi} \gamma^\beta \underline{A}_\beta \psi)_{x_2}]$$

$$(2.7) \quad S_D^{(2)} = -\frac{e^2}{2!} \int d^4x_1 d^4x_2 \{N[(\bar{\psi} \gamma^\alpha \underline{A}_\alpha \psi)_{x_1} (\bar{\psi} \gamma^\beta \underline{A}_\beta \psi)_{x_2}] \\ + N[(\bar{\psi} \gamma^\alpha \underline{A}_\alpha \psi)_{x_1} (\bar{\psi} \gamma^\beta \underline{A}_\beta \psi)_{x_2}]\}$$

$$(2.8) \quad S_E^{(2)} = -\frac{e^2}{2!} \int d^4x_1 d^4x_2 N[(\bar{\psi} \underline{A} \psi)_{x_1} (\bar{\psi} \underline{A} \psi)_{x_2}]$$

$$(2.9) \quad S_F^{(2)} = -\frac{e^2}{2!} \int d^4x_1 d^4x_2 N[(\bar{\psi} \gamma^\alpha \underline{A}_\alpha \psi)_{x_1} (\bar{\psi} \gamma^\beta \underline{A}_\beta \psi)_{x_2}].$$

We will go through each of these terms one by one to take note of their properties.

To start, we observe that Equation (2.4) does not involve any contractions. Intuitively, this implies that the process it describes consists of two basic processes as we saw from $S^{(1)}$ occurring independently of one another. Since we have already argued that such processes are unphysical, it follows that $S_A^{(2)}$ also leads to unphysical processes.

It turns out that the two terms in Equation (2.5) are actually identical under interchange of integration variables. We can therefore rewrite $S_B^{(2)}$ as

$$(2.10) \quad S_B^{(2)} = -e^2 \int d^4x_1 d^4x_2 N[(\bar{\psi} \underline{A} \psi)_{x_1} (\bar{\psi} \underline{A} \psi)_{x_2}].$$

This expression contains a fermion contraction. Hence we see a virtual fermion in the processes described by $S_B^{(2)}$; in particular, we see a virtual fermion propagating from x_2 to x_1 . The two uncontracted fermion and photon operator pairs deal with the creation and absorption of external particles.

As an example of a process under the purview of $S_B^{(2)}$, consider Compton scattering, whereby a photon is scattered by an electron ($\gamma + e^- \rightarrow \gamma + e^-$). We will pick out the initial electron absorption operator $\psi^+(x_2)$ from $\psi(x_2)$ and the final electron emission operator $\bar{\psi}^-(x_1)$ from $\bar{\psi}(x_1)$. However, note that either external photon can be emitted or absorbed at either point, and so we can choose between having the initial photon absorbed at x_2 or x_1 . In the former case, we take the $\underline{A}^+(x_2)$ term from $\underline{A}(x_2)$ to absorb the initial photon and the $\underline{A}^-(x_1)$ term from $\underline{A}(x_1)$ to emit the final photon. The resulting S -matrix term is given by

$$(2.11) \quad S_a = -e^2 \int d^4x_1 d^4x_2 \bar{\psi}^-(x_1) \gamma^\alpha \psi(x_1) \bar{\psi}(x_2) \gamma^\beta \underline{A}_\alpha^-(x_1) \underline{A}_\beta^+(x_2) \psi^+(x_2) \\ = -e^2 \int d^4x_1 d^4x_2 \bar{\psi}^-(x_1) \gamma^\alpha i S_F(x_1 - x_2) \gamma^\beta \underline{A}_\alpha^-(x_1) \underline{A}_\beta^+(x_2) \psi^+(x_2),$$

where S_F is the fermion propagator. The Feynman diagram for this process is given in Figure 2. The S -matrix term for the other case, where the initial photon is absorbed at x_1 , is constructed analogously, but we omit it here. There are other

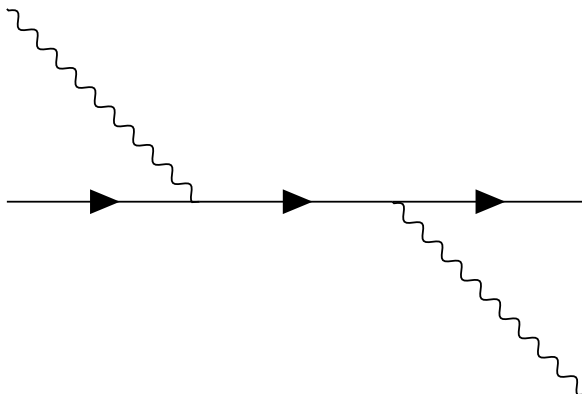


FIGURE 2. A Feynman diagram for Compton scattering where the initial photon is absorbed at x_2 .

processes described by $S_B^{(2)}$, but we will not show them here for the sake of saving space.

In $S_C^{(2)}$, we have photon-photon contraction and four uncontracted fermion operators. This term therefore gives rise to fermion-fermion scattering. As an example, consider Møller scattering ($e^- + e^- \rightarrow e^- + e^-$). Taking the electron parts of the absorption and creation operators in $S_C^{(2)}$, we get

$$(2.12) \quad S^{(2)}(2e^- \rightarrow 2e^-) = -\frac{e^2}{2} \int d^4x_1 d^4x_2 N[(\bar{\psi}^- \gamma^\alpha \psi^+)_{x_1} (\bar{\psi}^- \gamma^\beta \psi^+)_{x_2} iD_{F\alpha\beta}(x_1 - x_2)],$$

where $D_{F\alpha\beta}$ is the photon propagator. As with the photons of Compton scattering, either initial electron can be absorbed by either ψ^+ operator, and either final electron can be created by either $\bar{\psi}^-$, and so we see that four terms arise from this integral. But each of these terms differs from another only by the exchange of integration variables. Intuitively, this is equivalent to saying that an electron emitted and absorbed at x_1 is the same as one emitted and absorbed at x_2 . We are therefore left with a pair of integrals, which we multiply by a factor of 2. This cancels the factor of $\frac{1}{2}$.

In general, the n integration variables of the n^{th} order term $S^{(n)}$ in the S -matrix expansion can be attached to the n vertices of a Feynman diagram in $n!$ ways, and so the $\frac{1}{n!}$ that we see in $S^{(n)}$ can be ignored when considering topologically distinct Feynman diagrams.

We label the two initial electrons 1, 2 and the two final electrons 1', 2'. We are considering contributions to the transition

$$(2.13) \quad |i\rangle = c^\dagger(2)c^\dagger(1)|0\rangle \rightarrow |f\rangle = c^\dagger(2')c^\dagger(1')|0\rangle.$$

Let

$$(2.14) \quad \psi_j^+(x) = c(j)f_j(x), \quad \bar{\psi}_j^-(x) = c^\dagger(j)g_j(x)$$

be the parts of ψ^+ and $\bar{\psi}^-$ proportional to $c(j)$ and $c^\dagger(j)$, respectively, for $j = 1, 2, 1', 2'$. Then the two aforementioned distinguishable integral terms of $S^{(2)}(2e^- \rightarrow$

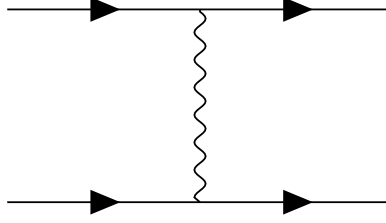


FIGURE 3. A Feynman diagram for Møller scattering where electron 2' is emitted at x_2 .

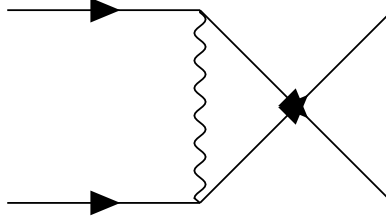


FIGURE 4. A Feynman diagram for Møller scattering where electron 1' is emitted at x_2 .

$2e^-$) are

$$(2.15) \quad S_a = -e^2 \int d^4x_1 d^4x_2 N[(\bar{\psi}_1^- \gamma^\alpha \psi_1^+)_{x_1} (\bar{\psi}_2^- \gamma^\beta \psi_2^+)_{x_2}] iD_{F\alpha\beta}(x_1 - x_2)$$

$$(2.16) \quad S_b = -e^2 \int d^4x_1 d^4x_2 N[(\bar{\psi}_2^- \gamma^\alpha \psi_1^+)_{x_1} (\bar{\psi}_1^- \gamma^\beta \psi_2^+)_{x_2}] iD_{F\alpha\beta}(x_1 - x_2).$$

It can be seen that S_a corresponds to the case where electron 2' is created at x_2 and S_b corresponds to the case where electron 1' is created at x_2 . The corresponding Feynman diagrams are shown in Figures 3 and 4.

Let us move on to $S_D^{(2)}$, which has two uncontracted fermion terms. As with $S_B^{(2)}$, the two terms are equal, and so we can remove the $\frac{1}{2}$ scalar. There are two possibilities for the external fermion: an electron or a positron. In the case of an electron, $S_D^{(2)}$ modifies the bare electron into a physical electron, or one surrounded by its photon cloud, by its interaction with the radiating field. This interaction changes the energy and therefore mass of the physical electron compared to the bare electron. This is called the self-energy of the electron. The integral for it diverges, but by adding the effects of the self-energy into the physical electron, we can renormalize this S -matrix term into a convergent solution. A Feynman diagram for the electron self-energy is shown in Figure 5.

Similarly, $S_E^{(2)}$ has two uncontracted photon operators, so it describes a photon self-energy. The photon self-energy is similarly divergent except under renormalization.

Note that $S_F^{(2)}$ has no uncontracted terms, no external particles, and therefore no transitions. It generates a vacuum diagram, shown in Figure 6.

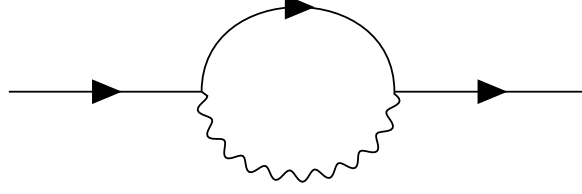


FIGURE 5. A Feynman diagram for the self-energy of a physical electron.

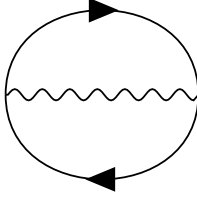


FIGURE 6. A vacuum Feynman diagram.

3. MANUALLY CALCULATING TRANSITION AMPLITUDES

It can be shown that interpreting Feynman diagrams as graphs in momentum space allows us to write down the matrix elements $\langle f|S^{(n)}|i\rangle$. Before we use this fact, we will calculate a specific example explicitly. Although the process is unphysical, consider the term of $S^{(1)}$ corresponding to electron scattering with emission of a photon ($e^- \rightarrow e^- + \gamma$), the same process as in Figure 1. This is the transition

$$(3.1) \quad |i\rangle = c^\dagger(\mathbf{p})|0\rangle \rightarrow |f\rangle = c^\dagger(\mathbf{p}')a^\dagger(\mathbf{k}')|0\rangle,$$

for an initial electron with momentum \mathbf{p} , final electron with \mathbf{p}' , and final photon with \mathbf{k}' . Using Equations 7.23 - 7.26 in the book, we get

$$(3.2) \quad \begin{aligned} \langle f|S^{(1)}|i\rangle &= \langle e^-\mathbf{p}'; \gamma\mathbf{k}' | ie \int d^4x \bar{\psi}^-(x) \gamma^\alpha A_\alpha^-(x) \psi^+(x) | e^-\mathbf{p} \rangle \\ &= ie \int d^4x \left[\left(\frac{m}{VE_{\mathbf{p}'}} \right)^{\frac{1}{2}} \bar{u}(\mathbf{p}') e^{ip'x} \right] \gamma^\alpha \left[\left(\frac{1}{2V\omega_{\mathbf{k}'}} \right)^{\frac{1}{2}} \epsilon_\alpha(\mathbf{k}') e^{ik'x} \right] \left[\left(\frac{m}{VE_{\mathbf{p}}} \right)^{\frac{1}{2}} u(\mathbf{p}) e^{-ipx} \right]. \end{aligned}$$

Using the fact that

$$\int d^4x e^{ix(p'+k'-p)} = (2\pi)^4 \delta^4(p' + k' - p),$$

we have

$$(3.3) \quad \langle f|S^{(1)}|i\rangle = [(2\pi)^4 \delta^4(p' + k' - p)] \left(\frac{m}{VE_{\mathbf{p}'}} \right)^{\frac{1}{2}} \left(\frac{m}{VE_{\mathbf{p}'}} \right)^{\frac{1}{2}} \left(\frac{1}{2V\omega_{\mathbf{k}'}} \right)^{\frac{1}{2}} \mathcal{M},$$

where

$$(3.4) \quad \mathcal{M} = ie \bar{u}(\mathbf{p}') \not{\epsilon}(\mathbf{k}' = \mathbf{p} - \mathbf{p}') u(\mathbf{p})$$

is called the Feynman amplitude for this process. Observe that the δ -function ensures conservation of energy and momentum at the vertex.

4. THE FEYNMAN RULES

Rather than using Equations 7.23-7.26 to calculate the transition amplitudes of each process, it is possible to simply write down the answer just from knowledge of the Feynman diagrams. In general, for the transition $|i\rangle \rightarrow |f\rangle$, we have that

$$(4.1) \quad \langle f|S|i\rangle = \delta_{fi} + [(2\pi)^4 \delta^4(P_f - P_i) \prod_{\text{ext.}} \left(\frac{m}{VE}\right)^{\frac{1}{2}} \prod_{\text{ext.}} \left(\frac{1}{2V\omega}\right)^{\frac{1}{2}}] \mathcal{M},$$

where P_f and P_i are the total four-momenta of the final and initial states, and the products iterate over all external fermions and photons, respectively. The total Feynman amplitude is

$$(4.2) \quad \mathcal{M} = \sum_{n=1}^{\infty} \mathcal{M}^{(n)},$$

where the $\mathcal{M}^{(n)}$ come from $S^{(n)}$. In particular, each $\mathcal{M}^{(n)}$ can be obtained by drawing all topologically distinct Feynman diagrams which contain n vertices and the correct external particles. The rules for writing $\mathcal{M}^{(n)}$ are written below, almost exactly as they appear in the book:

- (1) For each vertex, write a factor of $ie\gamma^\alpha$.
- (2) For each virtual photon, labelled with momentum k , write a factor of

$$iD_{F\alpha\beta}(k) = -i \frac{g_{\alpha\beta}}{k^2 + i\epsilon}.$$

- (3) For each virtual fermion, labelled with momentum p , write a factor of

$$iS_F(p) = i \frac{1}{\not{p} - m + i\epsilon}.$$

- (4) For each external particle, write one of these factors:

- (a) initial electron: $u_r(\mathbf{p})$
- (b) final electron: $\bar{u}_r(\mathbf{p})$
- (c) initial positron: $\bar{v}_r(\mathbf{p})$
- (d) final positron: $v_r(\mathbf{p})$
- (e) photon: $\epsilon_{r\alpha}(\mathbf{k})$

where \mathbf{p} or \mathbf{k} denote the particle's three-momentum and $r \in \{1, 2\}$ labels its spin or polarization state.

- (5) Write the spinor factors (γ , S_F , any four-spinors) for each fermion so that, reading from right to left, they occur in the same sequence as following the fermion line in the direction of its arrows.
- (6) For each closed fermion loop, multiply the trace by -1 .
- (7) For each four-momentum q which is not fixed by energy-momentum conservation, multiply by

$$\left(\frac{1}{2\pi}\right)^4 \int d^4q.$$

Note that this is necessary for each closed loop in the diagram.

- (8) Multiply the entire expression by -1 if an odd number of interchanges of fermion operators is required to set them in normal order.

These rules allow one to determine the transition amplitude of a given process using just the Feynman diagrams. It can be confirmed that using these rules on Figure 1 results in the same results as calculated manually, given by Equation (3.3).

5. CONCLUSIONS

Thus ends our very brief introduction to the utility of Feynman diagrams. If we were to suggest only one section from this paper to read, it would be Section 4. The tedium of referring to Equations 7.23-7.26 in the book can be altogether avoided by memorizing the eight rules outlined above. Section 4 therefore represents the culmination of the work in this paper, and indeed, many years of work by brilliant physicists. The reader can now be expected to easily compute the probability amplitudes of any arbitrary process in QED, which is an incredibly valuable tool.

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