Lecture: Resonance and Atomic-Field Interactions

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This lecture is devoted to the study of atomic resonance in the presence of an electromagnetic wave.

I. INTRODUCTION

The photo-electric effect and the reverse phenomenon are some of the first clues that light behaves as a particle, as a quanta of energy.

\[ eV_0 = \hbar \omega - \phi \]  

where \( \phi \) is the work function of the medium. It was shown that increasing the energy, intensity, of the light did not cause more energetic electrons, just more of the same energies. This effect has been explored very much over the years in the areas of photo-diodes, spectroscopy, LED’s, and resonant atomic systems. Optical resonance is a very interesting topic rooted in classical Electricity and Magnetism Theory and developed more fully in the quantum formalism.

II. THE THEORY OF RESONANCE

An atom in an electric field will act like a dipole. The electric field will push the electron and pull on the positively charged nucleus. The created dipole will then interact with the electric field, trying to align itself with the field. This will add an extra term to the Hamiltonian, due to the potential energy of the dipole in the field. This will add an extra term to the Hamiltonian and knowing that

\[ \mathbf{d} = \mathbf{d}_0 \mathbf{E}(t, \mathbf{r}) \]

or simplifying the equation, looking only in the \( x \) direction. We will assume the dipole will oscillate in the field and will approximate the dipole motion to that of a harmonic oscillator with natural frequency \( \omega_0 \) oscillating in the \( x \) direction. The new Hamiltonian will be:

\[ H = \frac{1}{2m} [p_x^2 + \omega_0^2 m^2 x^2] + \mathbf{d} \cdot \mathbf{E}(t, \mathbf{r}) \]  

or simplifying the equation, looking only in the \( x \) direction and knowing that \( \mathbf{d} = -e\mathbf{x} \), we get:

\[ H = \frac{1}{2m} [p_x^2 + \omega_0^2 m^2 x^2] - e x E_0(t, x) \]  

Using Hamilton’s equations (something from PHY 235):

\[ \dot{p}_x = -\frac{\partial H}{\partial x} \quad \text{and} \quad \dot{x} = \frac{\partial H}{\partial p_x} \]  

Solving these will lead to the equation of motion of a driven harmonic oscillator:

\[ \ddot{x} + \omega_0^2 x = \frac{e}{m} E(t, x) \]  

This will describe any driven harmonic oscillator. But this oscillator is also damped, due to the radiation released by accelerated charges. This energy decay rate can be derived using some classical EM theory and is given by

\[ \Gamma = \frac{2}{\tau_0} = \frac{2e\omega_0^2}{3mc^3} \]  

where \( \tau_0 \) is the characteristic decay time of the oscillator. In other words, if the electric field is turned off, the oscillator will stop moving after time \( \tau_0 \). Since the decay term is proportional to the velocity of the oscillating electron, it can be incorporated into the equation of motion by

\[ \ddot{x} + \Gamma \dot{x} + \omega_0^2 x = \frac{e}{m} E(t, x) \]  

Now, we can assume the electric field (a laser or other electromagnetic field) is oscillating at some frequency \( \omega \) with some amplitude \( E_0 \). The final equation of motion can be written as

\[ \ddot{x} + \Gamma \dot{x} + \omega_0^2 x = -\frac{eE_0}{m} e^{-i\omega t} \]  

To solve this second order equation, we need to make an assumption: that \( x = x e^{-i\omega t} \). This implies that the position of the electron is oscillating at the same frequency as the field. This is a fair assumption within the approximation that the internal force on the electron is small compared to the external electric field, i.e. the Hamiltonian for the hydrogen atom has little affect which is true for a strong external electric field. Subbing this solution into the equation of motion and and solving for \( x \) we obtain

\[ x = \frac{eE_0}{m} \frac{1}{\omega^2 - \omega_0^2 + i\omega \Gamma} \]  

This shows that the electron behaves as an oscillating particles with natural frequency \( \omega_0 \) and will experience resonance when the field frequency matches the natural frequency. The equation for \( x \) is generally simplified using the approximation that \( \omega_0 \gg \Gamma \). This will lead us to the equation:

\[ x \approx \frac{eE_0}{2m\omega_0^2} \frac{1}{\omega - \omega_0 + i\Gamma/2} \]  

The polarizability can be found using

\[ P = \frac{xe}{E_0} = \frac{e^2}{2m\omega_0^2} \frac{1}{\omega - \omega_0 + i\Gamma/2} \]
and the susceptibility is given by
\[
\chi(\omega) = \frac{NP}{\varepsilon_0} = \frac{Ne^2}{2m\omega_0\varepsilon_0} \frac{1}{\omega - \omega_0 + i\Gamma/2}.
\] (12)

As can be seen, \(\chi(\omega)\) can be separated into its real and imaginary parts and therefore, the index of refraction also can be separated into real and imaginary parts. The real part determines the dispersion, resulting in an index of refraction given by
\[
n(\omega) = 1 + \frac{Re\chi(\omega)}{2} \quad (13)
\]
and the imaginary part determines the absorption, with the absorption/gain coefficient given by
\[
\alpha(\omega) = \frac{4\pi\omega}{c} Im\chi(\omega)
\] (14)
and this can be positive or negative, corresponding to a gain or absorption, but is typically negative, corresponding to absorption.

### III. TREATING THIS PROBLEM IN QUANTUM MECHANICS

We will use the same Hamiltonian but rewrite it in terms of the atomic and interaction energies:
\[
H = H_A + H_I
\] (15)
Where the \(H_A\) is the typical atomic hamiltonian and the \(H_I\) is given by
\[
H_I = \langle \hat{d} \cdot \vec{E} \rangle
\] (16)
and for a time varying field in the \(x\) direction, this simplifies to
\[
H_I = \langle \hat{d}_x E \rangle = \langle e_x E \rangle = eE_0 e^{-i\omega t} \langle m|x|n \rangle
\] (17)

As you determined in an earlier homework, \(\langle m|x|n \rangle\) produces non-zero terms when \(m \neq n\). It actually only allows for \(m = n \pm 1\) which mean that there are transitions from state \(n\) to state \(m\). So, classically, there is the possibility of exciting an electron to a higher orbit, a higher oscillator state but only for higher harmonics in the driving frequency resonance.

The quantum description allows for state transitions anytime there is an external field and the pump frequency is close to the transition energy. This can be seen in an example of the Eq. (17). Looking at the states of Hydrogen, I will not go through the exact integral here of the transition probability, but it can be shown that the integral is nonzero.

### IV. APPLICATIONS OF RESONANCE

Lasers (also classical wave resonance)
Spectroscopy
Diodes and Photocells