Let \( x = A \sin(\omega t + \phi) \)
\[
\frac{dx}{dt} = A\omega \cos(\omega t + \phi)
\]
\[
\frac{d^2x}{dt^2} = -A\omega^2 \sin(\omega t + \phi)
\]

Substitute into differential eqn:
\[-A\omega^2 \sin(\omega t + \phi) + \frac{k}{m} A \sin(\omega t + \phi) = 0\]

True if \( \omega^2 = \frac{k}{m} \)!

So our little differential eqn has solns either
\[ x(t) = A \cos(\omega t + \phi) \quad \text{or} \quad x(t) = A \sin(\omega t + \phi) \]

Does it make sense?

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Harmonic Functions

\( \Rightarrow \) why we call it Simple Harmonic Motion!

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Let mass on spring oscillate on frictionless horizontal plane

View from above

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\( \rightarrow x \quad \rightarrow x_0 \)

\( \rightarrow \text{Time} \)
\[ F = -kx \quad x_0 = 0 \]
\[ ma = -kx \]
\[ m \frac{d^2x}{dt^2} = -kx \]
\[ \frac{d^2x}{dt^2} + \frac{k}{m} x = 0 \]

Equation of Motion for a Simple Harmonic Oscillator

Solve this for \( x(t) \) tells where spring is at any time.

This is what you need usually.

Let \( x = A \cos(\omega t + \phi) \)

\[ \text{Amplitude} \quad \omega \quad \text{Frequency} \]

\[ \frac{dx}{dt} = -A\omega \sin(\omega t + \phi) \]

\[ \frac{d^2x}{dt^2} = -A\omega^2 \cos(\omega t + \phi) \]

Substitute into differential eqn

\[ -A\omega^2 \cos(\omega t + \phi) + \frac{k}{m} A \cos(\omega t + \phi) = 0 \]

True if \( \omega^2 = \frac{k}{m} \) or \( \omega = \pm \sqrt{\frac{k}{m}} \)
The motion is periodic and repeats in Time T, called the Period

\[ \text{Frequency} \equiv \frac{1}{T} \text{ units of } \frac{1}{s} \text{ or Hertz, Hz} \]

\[ x(t) = A \cos(\omega t + \phi) \]

- Initial phase angle \( \phi \)
- Amplitude of motion \( A \)
- Frequency \( \omega = \frac{2\pi}{T} \) in radians

Circular motion can be thought of as a superposition of linear SHM in 2 dimensions

\[ A \cos \theta = y \]
\[ A \sin \theta = x \]

but \( \theta = \omega t \)

\[ x(t) = A \sin \omega t \]
\[ y(t) = A \cos \omega t \]
At maximum amplitude ... know all E is PE

\[ E = \frac{1}{2} kx^2 = \frac{1}{2} kA^2 \]

Total energy in a SHO is \( \frac{1}{2} kA^2 \)

**Example**

Show a mass oscillating on a vertical spring executed simple harmonic motion.

\[ F_{eq} = -ky \]

Equilibrium length (natural) of spring \( L \) no mass

\[ y_0 \]

Length \( L \) mass in equl:

\[ y' \]

Oscillates about \( y = y_0 \)

\[ EFy = 0 \Rightarrow k y_0 = mg \]

\[ y_0 = \frac{mg}{k} \]

\[ EFy = -k y + mg = ma = m \frac{d^2y}{dt^2} \]

Let \( y \to y' + \frac{mg}{k} \)

\[ \frac{dy}{dt} = \frac{dy'}{dt} \quad \text{and} \quad \frac{d^2y}{dt^2} = \frac{d^2y'}{dt^2} \]

\[ EFy \ \text{becomes} \]

\[ \frac{-k y'}{dt^2} = \frac{md^2y'}{dt^2} \]

This is eqn of motion for SHO we know how to solve!
The simple harmonic motion of a mass oscillates about $y_0$.

$$F = -ky'$$

where $y' = y - \frac{mg}{k}$

$$y'(t) = A \cos(\omega t + \theta_0)$$

with

$$\omega = \sqrt{\frac{k}{m}}$$

Gravity shifts the equilibrium point of motion from $y=0$ to $y'=0$.

All other results are the same as for SHM in variable $y'$.

**Example:** Simple Pendulum

Restoring force $F = mg \sin \theta$

**But** $\sin \theta \approx \theta$ for small $\theta$
Taylor's series expansion about $x=a$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \cdots + \frac{f^{(n-1)}(a)(x-a)^{n-1}}{(n-1)!} + \cdots$$

Look at:

$f(a) = \sin \theta$ about $\theta = 0$

$$\sin \theta = \sin(0) + \cos(0)(\theta - 0) + \frac{-\sin(0)(\theta - 0)^2}{2!} + \frac{[-\cos(0)](\theta - 0)^3}{3!} + \cdots$$

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots$$

for small $\theta$

$\sin \theta \approx \theta$

Look at small $\theta$

$\sin \theta \approx \theta$
Taylor Series expansion of $\cos x$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$\cos x \approx 1$ for $x$ small

This is a very useful tool in Physics!!

Back to simple pendulum

$$s = L\theta$$

$$m \frac{d^2 s}{dt^2} = -mg\sin \theta \approx -mg\theta = -mg\frac{s}{L}$$

Some form of SHO differential eqn

$$\frac{d^2 s}{dt^2} + \frac{9}{L} s = 0$$

$$s(t) = A \cos (\omega t + \phi) \quad \text{where} \quad \omega = \sqrt{\frac{9}{L}}$$
Mass attached to spring oscillates back and forth as shown in position-time graph below.

At point P, the mass has:

1. Positive velocity, positive acceleration.
2. Positive velocity, negative acceleration.
3. Negative velocity, positive acceleration.
4. Negative velocity, negative acceleration.
5. Zero velocity but acceleration.
6. Some velocity, but zero acceleration.
7. Zero velocity, zero acceleration.
A simple pendulum has a period $T$ on Earth. How does this compare to the period of the same pendulum if taken to the moon?

1. $T_{\text{Earth}} > T_{\text{Moon}}$
2. $T_{\text{Earth}} < T_{\text{Moon}}$
3. $T_{\text{Earth}} = T_{\text{Moon}}$
\[ \omega = \sqrt{g_R} \]

\[ \omega = \frac{2\pi}{T} \]

\[ T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{L}{g}} \]

\[ g_{\text{moon}} < g_{\text{earth}} \]

\[ \therefore T_{\text{moon}} > T_{\text{earth}} \rightarrow (2) \]
Theoretical Chemistry Example

quantized

vibrational energy levels

Diatomic Molecule
or
Atomic Bond

Given \( F = -\frac{C}{r^2} + \frac{D}{r^3} \)

What is frequency and period of vibration?

What is equilibrium position = \( r_0 \)

at \( r_0 = r \), \( F = 0 \)

\[ \frac{C}{r_0^2} = \frac{D}{r_0^3} \quad \Rightarrow \quad r_0 = \frac{D}{C} \]

Consider small oscillations about \( r_0 \)

\( r = r_0 + x \)

\[ F = -\frac{C}{(r_0+x)^2} + \frac{D}{(r_0+x)^3} \]

recall from Calculus \( \Rightarrow \) Taylor's series expansions

\[ f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \ldots + \frac{f^{(n-1)}(a)(x-a)^{n-1}}{(n-1)!} \]
Thus

\[(1 + x)^{-2} = 1 - 2x + 3x^2 + \ldots\]

\[(1 + x)^{-3} = 1 - 3x + 6x^2 + \ldots\]

\[F = -\frac{C}{(r_0 + x)^2} + \frac{D}{(r_0 + x)^3} = -\frac{C}{r_0^2 (1 + \frac{x}{r_0})^2} + \frac{D}{r_0^3 (1 + \frac{x}{r_0})^3}\]

\[x \text{ is small} \]

\[\Rightarrow x \approx x\]

Sub in Taylor's series expansion, drop terms \(O\left(\frac{x}{r_0}\right)^2\) or smaller.

\[F = -\frac{C}{r_0^2} (1 - 2\frac{x}{r_0}) + \frac{D}{r_0^3} (1 - 3\frac{x}{r_0})\]

\[F = -\frac{C}{r_0^2} + \frac{D}{r_0^3} + C \cdot 2 \cdot \frac{x}{r_0} - D \cdot 3 \cdot \frac{x}{r_0}\]

\[\text{by def of equilibrium position } \Rightarrow \text{ as shown earlier}\]

\[m \frac{d^2x}{dt^2} = -\frac{1}{\gamma} (3D + 2C) x\]

\[\frac{dx}{dt} + \frac{x}{M} x = 0 \quad \text{SHM with} \quad \omega = \sqrt{\frac{x}{M}}\]

\[\text{Period} = \frac{2\pi}{\omega}\]