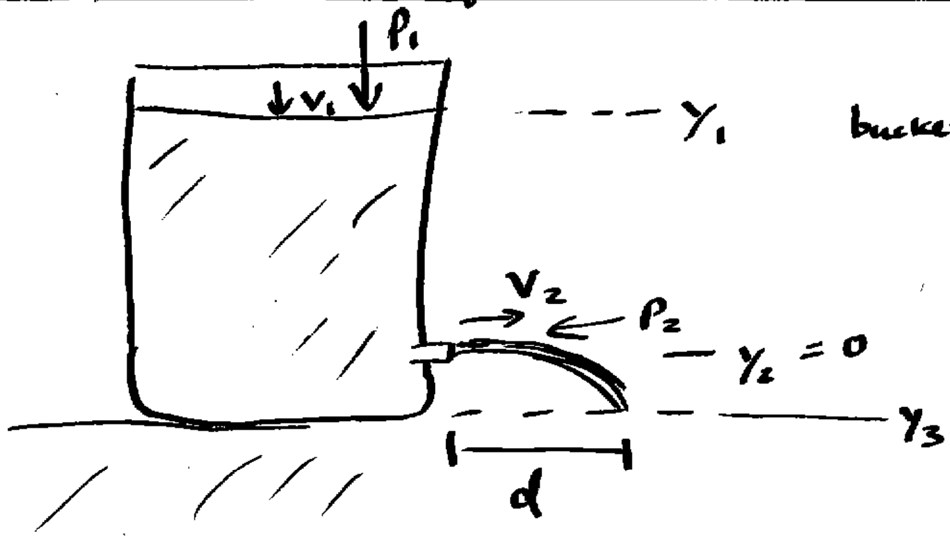
Bernoulli's eqn example



bucket of water

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g h = P_2 + \frac{1}{2}\rho v_2^2$$

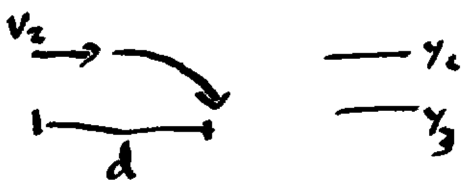
$$P_1 \sim P_2 = P_{ATM}$$

$$h = y_1 - y_2$$

$$v_1 \approx 0$$

$$\therefore \quad \cancel{\rho} g h = \frac{1}{2}\cancel{\rho} v_2^2$$

$$v_2 = \sqrt{2gh}$$

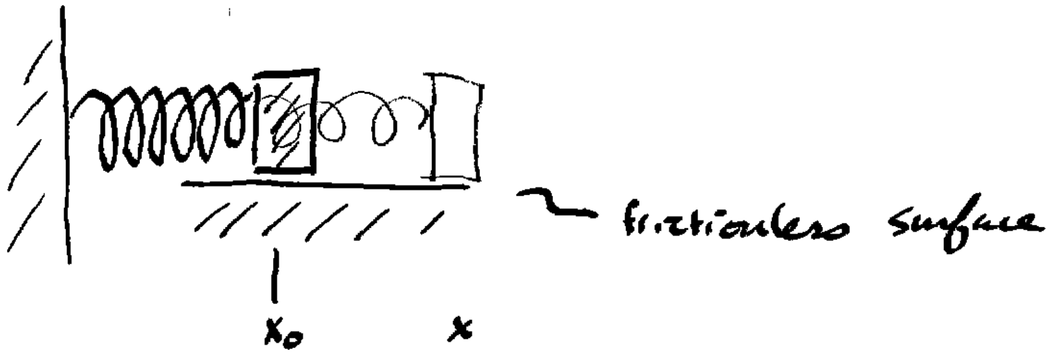


Standard Projectile Problem to find d

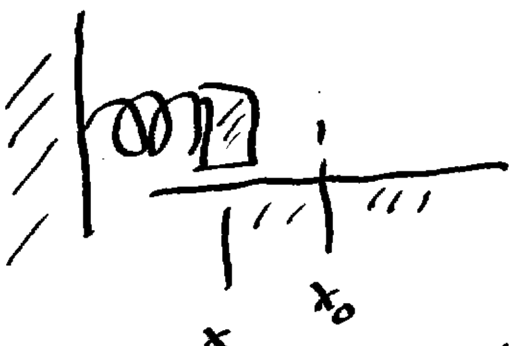
$$y_3 = y_2 + \underbrace{v_{0y}}_{=0} t + \frac{1}{2} \overbrace{a}^{-9.8} t^2 \Rightarrow t$$

$$d = v_2 t \quad \text{subst. in}$$

Simple Harmonic Motion (SHM)

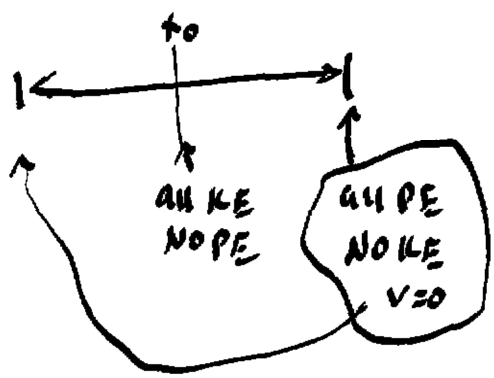


$\leftarrow \vec{F} = -k(x - x_0)$



$\rightarrow \vec{F} = -k(x - x_0)$

goes back + forth



Energy flow

$KE \rightarrow PE \rightarrow KE$

$\frac{1}{2}mv^2 \quad \frac{1}{2}k(x-x_0)^2$

$F = -kx \qquad x_0 = 0$

$ma = -kx$

$m\frac{d^2x}{dt^2} = -kx$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

differential equation
equation of Motion for
a Simple Harmonic
Oscillator

Solve this for $X(t)$ → tells where spring is
as a fn of time
This is what you need usually

Let $x = A\cos(\omega t + \phi)$

Amplitude, frequency, initial phase — all just constants for now

general soln ... constants set by "initial" and "boundary" conditions specific to problem

$$\frac{dx}{dt} = -A\omega\sin(\omega t + \phi)$$

$$\frac{d^2x}{dt^2} = -A\omega^2 \cos(\omega t + \phi)$$

Substitute into differential eqn

$$-A\omega^2\cos(\omega t+\phi) + \frac{k}{M}A\cos(\omega t+\phi) = 0$$

True $\underline{\underline{\text{if}}}$ $\omega^2 = k/M$ or $\omega = \pm\sqrt{k/M}$

Let $x = A\sin(\omega t+\phi)$

$$\frac{dx}{dt} = A\omega\cos(\omega t+\phi)$$

$$\frac{d^2x}{dt^2} = -A\omega^2\sin(\omega t+\phi)$$

substitute into differential eqn —

$$-A\omega^2\sin(\omega t+\phi) + \frac{k}{M}A\sin(\omega t+\phi) = 0$$

True if $\omega^2 = k/M$!

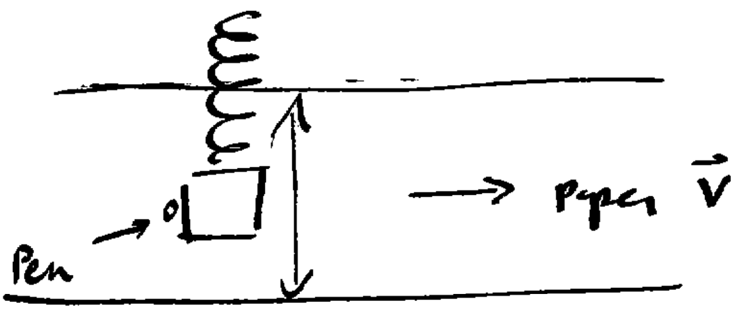
So our little differential eqn has solns

either $x(t) = A\cos(\omega t+\phi)$ OR $x(t) = A\sin(\omega t+\phi)$

Does it make sense?

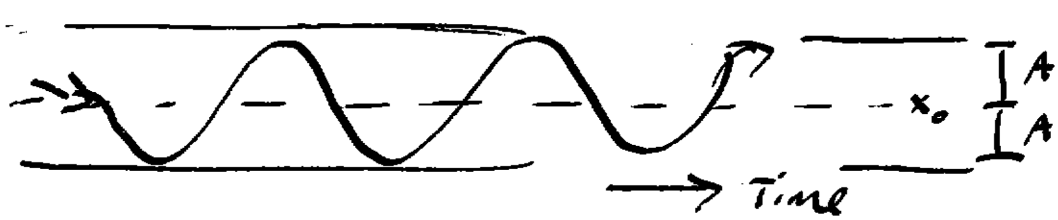
Harmonic Functions

⇒ why we call it Simple Harmonic Motion!



let mass on spring oscillate on frictionless horizontal plane

View from above



The motion is Periodic

repeats in Time T, called the Period

Frequency $\equiv \frac{1}{T}$ units of $\frac{1}{s}$ or Hertz, Hz

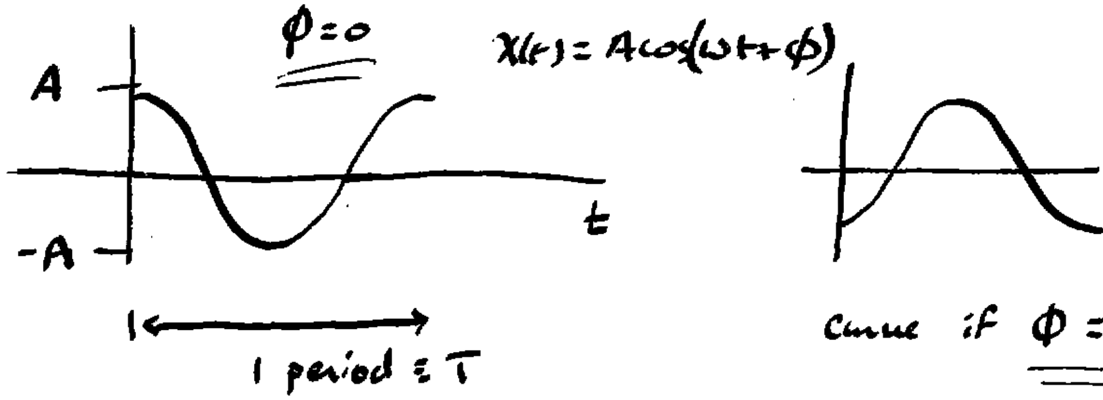
$x(t) = A\cos(\omega t + \phi)$

ϕ ← initial phase angle

ω ← frequency $\omega = \frac{2\pi}{T}$ in radians

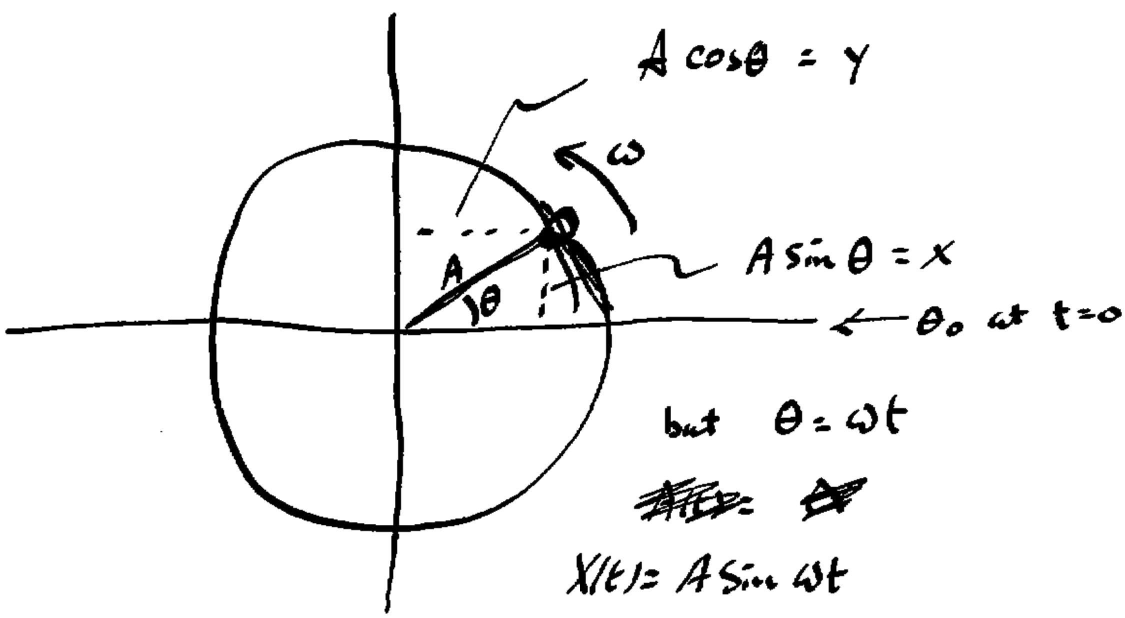
A ↑ Amplitude of Motion

$\phi = 0$ $\quad x(t) = A\cos(\omega t + \phi)$



Curve if $\phi = \pi$

Circular motion can be thought of as a Superposition of linear SHM in 2 dimensions



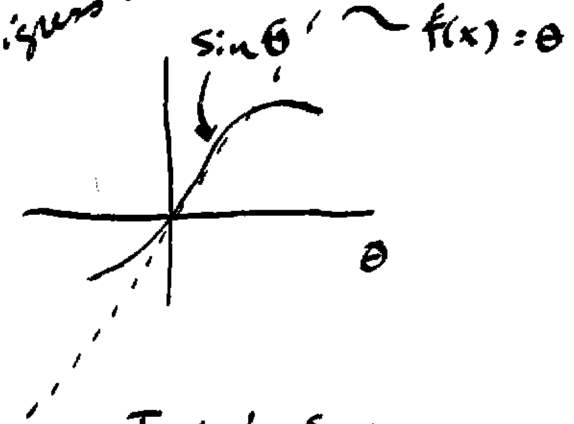
but $\theta = \omega t$

$X(t) = A\sin\omega t$

$Y(t) = A\cos\omega t$

if diff θ_0 ... put in init. phase angle into argument

Digress a little



Look at small θ

$$\sin\theta \approx \theta$$

Taylor's Series expansion about $x=a$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \dots + \frac{f^{(n-1)}(a)(x-a)^{n-1}}{(n-1)!} + \dots$$

Look at

$$f(x) = \sin\theta \quad \text{about } \theta = 0$$

$$\sin\theta = \sin(0) + \cos(0)(\theta-0) + \frac{[-\sin(0)](\theta-0)^2}{2!} + \frac{[-\cos(0)](\theta-0)^3}{3!} + \dots$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \dots$$

for small θ

$$\sin\theta \approx \theta$$

Error in our Approximation

θ_{deg}	θ_{rad}	$\sin\theta$	$\frac{\Delta}{\sin\theta}$
1.0	0.017453	0.017452	.006%
10	0.1745	0.1736	0.51%
30			4.7%

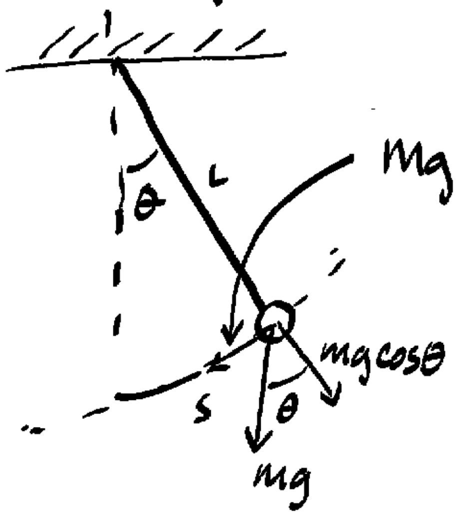
Taylor's Series expansion of $\cos x$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$\cos x \approx 1$ for x small

This is a Very useful Tool in Physics !!

back To Simple Pendulum



$\cancel{\frac{d\sin\theta}{d\theta} = \cos\theta}$

$$s = L\theta$$

$$m\frac{d^2s}{dt^2} = -mg\sin\theta \approx -mg\theta = -mg\,{}^{s}/_{L}$$

$\cancel{m\frac{d^2(L\theta)}{dt^2} \approx -mg\theta}$

$$\boxed{\frac{d^2s}{dt^2} + \frac{g}{L}s = 0}$$

$\cancel{\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta}$

Same form as SHO differential eqn

$\therefore$

$$s(t) = A\cos(\omega t + \phi) \quad \text{where}$$

$$\omega = \sqrt{g/L}$$

Tells us ~~spring~~ mass oscillates w/ SHM about y_0

$$F = -k y' \qquad \text{where } y' = y - \frac{mg}{k}$$

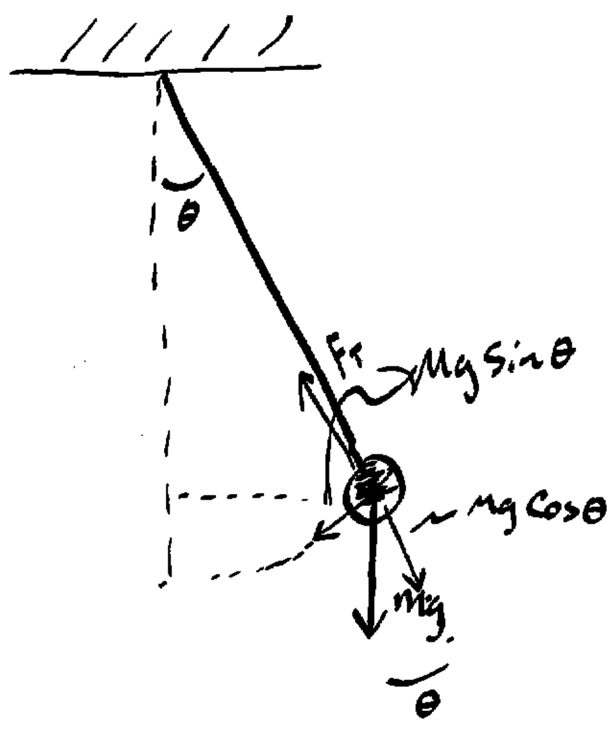
$$y'(t) = A\cos(\omega t + \theta_0)$$

w/ $\omega = \sqrt{\frac{k}{m}}$

gravity shifts the equilibrium point of motion from $y=0$ to $y'=0$

All other results are the same as for SHM in "variable y'"!

<u>example</u> Simple Pendulum



restoring force $F = mg\sin\theta$

<u>But</u> $\sin\theta \approx \theta$ for small θ

→ Whoa! Where did I get this Magic?!

Spring

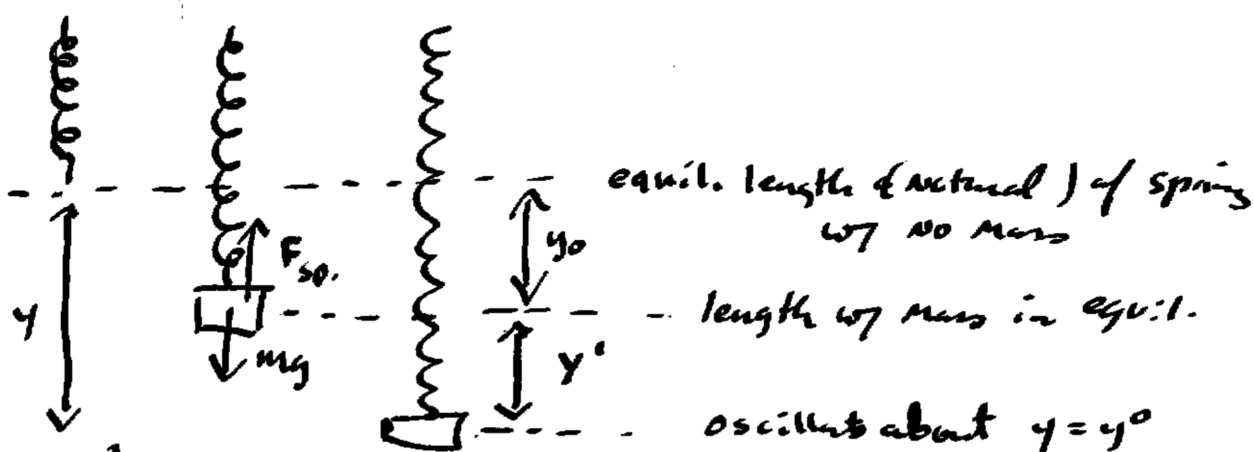
AT Maximum Amplitude ... know all E is PE

$$E = \frac{1}{2} k x^2 = \frac{1}{2} k A^2$$

$\therefore$ $\boxed{\text{Total Energy in a SHO is } \tfrac{1}{2} k A^2}$

Example

Show a mass oscillating on a vertical spring executes simple Harmonic Motion



$$\Sigma F_y = 0 \Rightarrow k y_0 = mg \qquad y_0 = mg/k$$

$$\Sigma F_y = -ky + mg = ma = m\frac{d^2y}{dt^2}$$

Let $y \rightarrow y' + \frac{mg}{k}$

$$\frac{dy}{dt} = \frac{dy'}{dt} \quad \text{and} \quad \frac{d^2y}{dt^2} = \frac{d^2y'}{dt^2}$$

So

ΣF_y becomes $\boxed{-ky' = m\frac{d^2y'}{dt^2}}$

This is eqn of Motion for SHO we know how to solve!