Bernoulli's eqn example

\[ P_1 + \frac{1}{2} \rho v_1^2 + \rho gh = P_2 + \frac{1}{2} \rho v_2^2 \]

\[ P_1 - P_2 = \rho_{atm} \]

\[ h = y_1 - y_2 \]

\[ v_1 \approx 0 \]

\[ \rho gh = \frac{1}{2} \rho v_2^2 \]

\[ v_2 = \sqrt{2gh} \]

\[ y_3 = y_2 + \frac{v_{oy} t}{2} + \frac{1}{2} a t^2 \]

\[ d = v_2 t \]

Standard Projectile Problem to find \( d \)
Simple Harmonic Motion (SHM)

\[ F = -k(x-x_0) \]

- Goes back and forth
- Energy flow:
  \[ KE \rightarrow PE \rightarrow KE \]
  \[ \frac{1}{2}mv^2 \rightarrow \frac{1}{2}k(x-x_0)^2 \]
\[ F = -kx \quad x_0 = 0 \]

\[ ma = -kx \]

\[ m \frac{d^2x}{dt^2} = -kx \]

\[ \frac{d^3x}{dt^2} + \frac{k}{m} x = 0 \]

**differential equation**

**equation of Motion for a Simple Harmonic Oscillator**

\( \Rightarrow \) Solve this for \( x(t) \) \( \Rightarrow \) tells where spring is

\( \Rightarrow \) at a fun of time

This is what you need usually

**Let** \[ x = A \cos(\omega t + \phi) \]

\( \uparrow \quad \downarrow \quad \) initial phase

\[ \text{Amplitude} \quad \text{Frequency} \]

\[ \frac{dx}{dt} = -A \omega \sin(\omega t + \phi) \]

\[ \frac{d^2x}{dt^2} = -A \omega^2 \cos(\omega t + \phi) \]

Substitute into differential eqn

\[ -A \omega^2 \cos(\omega t + \phi) + \frac{k}{m} A \cos(\omega t + \phi) = 0 \]

**True if** \( \omega^2 = \frac{k}{m} \) \( \Rightarrow \omega = \pm \sqrt{\frac{k}{m}} \)

\[ \Rightarrow \text{general soln} \quad \text{constants set by "initial" and "boundary" conditions specific to problem} \]
Let \( x = A \sin(\omega t + \phi) \)
\[
\frac{dx}{dt} = A \omega \cos(\omega t + \phi)
\]
\[
\frac{d^2x}{dt^2} = -A \omega^2 \sin(\omega t + \phi)
\]
substitute into differential eqn —
\[-A \omega^2 \sin(\omega t + \phi) + \frac{k}{m} A \sin(\omega t + \phi) = 0\]
True if \( \omega^2 = \frac{k}{m} \)!
So our little differential eqn has solns either \( x(t) = A \cos(\omega t + \phi) \) or \( x(t) = A \sin(\omega t + \phi) \)

Does it make sense?

Harmonic Functions

Why we call it Simple Harmonic Motion!

Let mass on spring oscillate on frictionless horizontal plane
View from above

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Pen ➔

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The motion is periodic. It repeats in time $T$, called the period.

Frequency $\hbar = \frac{1}{T}$ units of $\frac{1}{5}$ or Hertz, Hz.

$x(t) = A \cos(\omega t + \phi)$ initial phase angle

Amplitude of motion

$\omega = \frac{2\pi}{T}$ in radians.

$A \begin{array}{c}
\phi = 0 \\
\end{array}$

$x(t) = A \cos(\omega t + \phi)$

Curve: if $\phi = \pi$

Circular motion can be thought of as a superposition of linear SHM in 2 dimensions.

$A \cos \theta = y$

$A \sin \theta = x$

but $\theta = \omega t$

$X(t) = A \sin \omega t$

$Y(t) = A \cos \omega t$

if $\theta = \theta_0$ ... put in initial phase angle into argument.
Taylor's series expansion about $x = a$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \cdots + \frac{f^{(n-1)}(a)(x-a)^{n-1}}{(n-1)!} + \cdots$$

Look at:

$$f(x) = \sin \theta \quad \text{about} \quad \theta = 0$$

$$\sin \theta = \sin(0) + \cos(0)(\theta-0) + \frac{-\sin(0)}{2!}(\theta-0)^2 + \frac{-\cos(0)}{3!}(\theta-0)^3 + \ldots$$

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \ldots$$

For small $\theta$

$$\sin \theta \approx \theta \quad \theta \ll 1$$

<table>
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<tr>
<th>$\theta$</th>
<th>$\theta_{\sin}$</th>
<th>$\frac{\theta_{\sin}}{\sin \theta}$</th>
<th>$\frac{\theta}{\sin \theta}$</th>
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<tr>
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<td></td>
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</tbody>
</table>
Taylor series expansion of \( \cos x \)

\[
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots
\]

\( \cos x \approx 1 \) for \( x \) small

This is a very useful tool in physics!!

back to simple pendulum

\[
\begin{align*}
\theta & = \theta_0 \\
\frac{d^2\theta}{dt^2} & = -\frac{mg}{L} \sin \theta \\
& \approx -mg \theta = -mg \frac{\theta}{L}
\end{align*}
\]

\[
\frac{d^2\theta}{dt^2} + \frac{g}{L} \theta = 0
\]

\( \theta(t) = A \cos(\omega t + \phi) \) where

\[
\omega = \sqrt{\frac{g}{L}}
\]
Tells us spring mass oscillator w/ SHM about $y_0$

$F = -k y'$

where $y' = y - \frac{mg}{k}$

$y'(t) = A \cos(\omega t + \theta_0)$

$\omega = \sqrt{\frac{k}{m}}$

Gravity shifts the equilibrium point of motion from $y = 0$ to $y' = 0$

All other results are the same w/ for SHM in variable $y'$

Example: Simple Pendulum

Restoring force $F = mg \sin \theta$

But $\sin \theta \approx \theta$ for small $\theta$

What? Where did I get this magic?!
AT maximum amplitude ... know all E is PE

\[ E = \frac{1}{2} kx^2 = \frac{1}{2} kA^2 \]

Total Energy in a SHO is \( \frac{1}{2} kA^2 \)

**Example**

Show a mass oscillating on a vertical spring executed simple harmonic motion.

\[ y = y_0 - \frac{mg}{k} \]

\[ \Sigma F = 0 \Rightarrow k(y_0 - \frac{mg}{k}) - mg = ma = \frac{md^2y}{dt^2} \]

\[ \Sigma F_y = -ky + mg = ma = \frac{md^2y}{dt^2} \]

Let \( y \rightarrow y' + \frac{mg}{k} \)

\[ \frac{dy}{dt} = \frac{dy'}{dt} \quad \text{and} \quad \frac{d^2y}{dt^2} = \frac{d^2y'}{dt^2} \]

\( \Sigma F_y \) becomes \( -ky' = \frac{md^2y'}{dt^2} \)

This is eqn of motion for SHO we know how to solve!